Nonlinear Systems and Control
Lecture 3

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Master Course in Electronic and Communication Engineering
Credits (3/0/3)

Outline

1 Describing Function Concept
2 Describing Function Analysis
Describing Function Concept: Basic Idea

Goal
- Predict the existence of periodic solutions \( y(t) = y(t + 2\pi/\omega), \ t \in \mathbb{R}, \) and their oscillation frequency \( \omega \)

\[ \Rightarrow \text{Method of harmonic balance} \]

Starting Point

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &=Cx \\
u &= -\psi(y)
\end{align*}
\]

- \((A, B)\) is controllable
- \((A, C)\) is observable
- \(\psi : \mathbb{R} \rightarrow \mathbb{R}\) is time-invariant nonlinearity

Describing Function Concept: Harmonic Balance

Periodicity Assumption
- Assume periodic solution \( y \) represented as Fourier series

\[
y(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} a_k \sin k\omega t + b_k \cos k\omega t \quad (a_k, b_k \in \mathbb{R})
\]

- Conclude periodicity of \( \psi(y(t)) \) with same frequency

\[
\psi(y(t)) = \frac{d_0}{2} + \sum_{k=1}^{\infty} c_k \sin k\omega t + d_k \cos k\omega t \quad (c_k, d_k \in \mathbb{R})
\]

- Generally, \( c_k, d_k \) depend on all coefficients \( a_k, b_k \)

\[ \Rightarrow \text{Reduce investigation to first harmonic of input and output} \]
Describing Function Concept: Output of Nonlinearity

Computation of the First Harmonic

\[ \psi(a \sin \omega t) \approx \frac{d_0(a)}{2} + c_1(a) \sin \omega t + d_1(a) \cos \omega t \]

\[ = d_0(a) + \sqrt{c_1(a)^2 + d_1(a)^2} \sin(\omega t + \arctan \frac{d_1(a)}{c_1(a)}) \]

Fourier Coefficients

\[ d_0(a) = \frac{1}{2\pi} \int_0^{2\pi} \psi(a \sin \omega t) dt \]

\[ c_1(a) = \frac{1}{\pi} \int_0^{2\pi} \psi(a \sin \omega t) \sin \omega t d(\omega t) \]

\[ d_1(a) = \frac{1}{\pi} \int_0^{2\pi} \psi(a \sin \omega t) \cos \omega t d(\omega t) \]

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Describing Function Concept: Definition

Definition (Describing Function)

Consider the system in Equation (1) and assume that the nonlinearity \( \psi : \mathbb{R} \to \mathbb{R} \) is skew-symmetric (that is, \( d_0(a) = 0 \) in the above computation). The describing function \( N(a) : \mathbb{R} \to \mathbb{C} \) for \( \psi \) is defined as

\[ N(a) = \sqrt{c_1(a)^2 + d_1^2(a)} \frac{c_1(a)}{a} e^{j \arctan \frac{d_1(a)}{c_1(a)}} = \frac{c_1(a)}{a} + j \frac{d_1(a)}{a} \]

Discussion of Approximation

- Neglecting all terms in the Fourier series with \( k > 1 \)
- Underlying assumption: \( G(s) \) is low-pass filter that attenuates high-frequencies

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Describing Function Concept: Relay Example

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Describing Function Concept: Hysteresis Example

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Describing Function Analysis: System Analysis

Harmonic Balance Equation

- \( e(t) = a \sin \omega t \)
- \( v(t) = a|N(a)| \sin(\omega t + \angle N(a)) \)
- \( y(t) = |G(j\omega)||N(a)| \sin(\omega t + \angle N(a) + \angle G(j\omega)) \)

⇒ **Condition:** 1 = \(-G(j\omega)N(a)\) or \(G(j\omega) = -\frac{1}{N(a)}\)

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Describing Function Analysis: Analysis Example

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Describing Function Analysis: Stability

Possible Properties of Predicted Oscillations
- Decay
- Explode
- Sustained
⇒ Generalized Nyquist plot analysis

Theorem (Nyquist Criterion)
Assume a classical feedback loop with a constant gain $K$ and a transfer function $G$ with no poles on the $j\omega$-axis. Let $p$ be the number of poles of $G$ in the open right half plane and let $n$ be the number of times the $G(j\omega)$-locus encircles the point $-1/K + j0$ in the complex plane (anticlockwise). Then, all closed-loop poles are in the left half plane if and only if $p - n = 0$.

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Describing Function Analysis: Stability

Illustration

Generalized Nyquist Criterion
- Assume that $K \in \mathbb{C}$ (instead of $K \in \mathbb{R}$)
⇒ Nyquist criterion now applies for encirclement of the complex point $-1/K$

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Describing Function Analysis: Stability

Application of Extended Nyquist Criterion to Nonlinear System

Prediction

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Describing Function Analysis: Summary and Remarks

Summary

- Describing function $N(a)$: transfer function between the first harmonic of the input and the output of the nonlinearity
- Analysis: predict periodic solutions of nonlinear systems based on harmonic balance equation: $1 = -G(j\omega)N(a)$
- Approximation: focus on first harmonic of periodic solutions
- Stability: Prediction based on extended Nyquist analysis

Assumptions

- Skew-symmetric time-invariant nonlinearity
- Focus on first harmonic of periodic solutions ($G$ is low-pass filter)

Remarks

- Predictions can be wrong due to approximation