Lecture Notes on

Differential Equations

Emre Sermutlu
To my wife Nurten and my daughters İlayda and Alara
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Preface

This set of lecture notes for ordinary and partial differential equations grew out of the course Engineering Mathematics I have taught at Çankaya University since 1999. It is a one-semester course for second year students. The main audience for this text, of course, is students. Presentation is user-friendly. There are more examples and fewer theorems than usual.

The material is based on a solid background in calculus. The student is assumed to be familiar with algebra, trigonometry, functions and graphs, series, differentiation, and most importantly, integration techniques of various kinds. It is my (and my students’) sad experience that if you cannot differentiate and integrate, you cannot solve differential equations. Knowledge of Linear Algebra, except for the determinants of a simple nature, is not assumed.

There are 14 chapters. Each chapter can be covered in one week. After a summary of methods and solved exercises, there are a number of end of chapter exercises with answers. The exercises that take exceptionally longer times are marked with a star. (★) Nobody can learn how to solve problems by watching someone else solve problems. So I advise the students to try each problem on their own.

I would like to thank all my students who helped me write this book by the valuable feedback they provided. In particular, special thanks are for Nuh Coşkun, Nevrez İmamoğlu, Nilgün Dinçarslan and Işıl Leoloğlu who have made a very extensive and meticulous check of the whole manuscript.

You may send all kinds of comments, suggestions and error reports to sermutlu@cankaya.edu.tr.

Assist. Prof. Dr. Emre Sermutlu
Chapter 1

First Order Differential Equations

The subject of differential equations is an important part of applied mathematics. Many real life problems can be formulated as differential equations. In this chapter we will first learn the basic concepts and classification of differential equations, then we will see where they come from and how the simplest ones are solved. The concepts and techniques of calculus, especially integration, will be necessary to understand differential equations.

1.1 Definitions

**Ordinary Differential Equation:** An ordinary differential equation is an equation that contains derivatives of an unknown function \( y(x) \).

**Partial Differential Equation:** A partial differential equation is a differential equation involving an unknown function of two or more variables, like \( u(x, y) \).

For example,

\[
y'' - 4y' + y = 0
\]
\[
\sqrt{y^2 + 1} = x^2 y + \sin x
\]

are ordinary differential equations.

\[
u_{xx} + u_{yy} = 0
\]
\[
u_x^2 + u_y^2 = \ln u
\]
are partial differential equations. (Partial Differential Equations are usually much more difficult)

**Order:** The order of a differential equation is the order of the highest derivative that occurs in the equation.

A first order differential equation contains \( y', y \) and \( x \) so it is of the form 
\[
F(x, y, y') = 0 \text{ or } y' = f(x, y).
\]

For example, the following differential equations are first order:
\[
y' + x^2y = e^x \\
x'y' = (1 + y^2) \\
y'^2 = 4xy
\]

While these are second order:
\[
y'' - x^2y' + y = 1 + \sin x \\
y'' + 6yy' = x^3
\]

**General and Particular Solutions:** A general solution of a differential equation involves arbitrary constants. In a particular solution, these constants are determined using initial values.

As an example, consider the differential equation \( y' = 2x \).
\[y = x^2 + c\] is a general solution, 
\[y = x^2 + 4\] is a particular solution.

**Example 1.1** Find the general solution of the differential equation \( y'' = 0 \). Then find the particular solution that satisfies \( y(0) = 5, y'(0) = 3 \).

\[
y'' = 0 \quad \Rightarrow \quad y' = c \quad \Rightarrow \quad y = cx + d.
\]
This is the general solution.
\[
y'(0) = 3 \quad \Rightarrow \quad c = 3, \quad y(0) = 5 \quad \Rightarrow \quad d = 5
\]
Therefore \( y = 3x + 5 \). This is the particular solution.

**Explicit and Implicit Solutions:** \( y = f(x) \) is an explicit solution, 
\( F(x, y) = 0 \) is an implicit solution. We have to solve equations to obtain \( y \) for a given \( x \) in implicit solutions, whereas it is straightforward for explicit solutions.

For example, \( y = e^{4x} \) is an explicit solution of the equation \( y' = 4y \).
\[x^3 + y^3 = 1\] is an implicit solution of the equation \( y^2y' + x^2 = 0 \).
1.2 Mathematical Modeling

Differential equations are the natural tools to formulate, solve and understand many engineering and scientific systems. The mathematical models of most of the simple systems are differential equations.

Example 1.2 The rate of growth of a population is proportional to itself. Find the population as a function of time.

\[ \frac{dP}{dt} = \alpha P \]

\[ P = P_0 e^{\alpha t} \]

where \( P_0 = P(0) \)

Example 1.3 The downward acceleration of an object in free fall is \( g \). Find the height as a function of time if the initial height is \( y_0 \) and initial speed is \( v_0 \).

\[ \frac{d^2 y}{dt^2} = -g \]

\[ \frac{dy}{dt} = -gt + v_0 \]

\[ y = -\frac{1}{2} gt^2 + v_0 t + y_0 \]

1.3 Separable Equations

If we can separate \( x \) and \( y \) in a first order differential equation and put them to different sides as \( g(y)dy = f(x)dx \), it is called a separable equation. We can find the solution by integrating both sides. (Don’t forget the integration constant!)

\[ \int g(y)dy = \int f(x)dx + c \quad (1.1) \]
Example 1.4 Solve the initial value problem \( y' + y^2xe^x = 0 \), \( y(0) = 2 \)

\[
y' = -y^2xe^x \quad \Rightarrow \quad -\frac{dy}{y^2} = xe^x \, dx
\]

\[
- \int \frac{dy}{y^2} = \int xe^x \, dx
\]

Using integration by parts, we have \( u = x, dv = e^x \, dx, du = dx, v = e^x \)

Therefore

\[
\frac{1}{y} = xe^x - \int e^x \, dx \quad \Rightarrow \quad \frac{1}{y} = xe^x - e^x + c
\]

\[
y = \frac{1}{xe^x - e^x + c}
\]

This is the general solution. Now we will use the condition \( y(0) = 2 \) to determine the constant \( c \).

\[
2 = \frac{1}{-1 + c} \quad \Rightarrow \quad c = \frac{3}{2}
\]

\[
y = \frac{1}{xe^x - e^x + \frac{3}{2}}
\]

Example 1.5 Find the general solution of the differential equation \( y' + y^2 = 1 \).

\[
\frac{dy}{dx} + y^2 = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 1 - y^2 \quad \Rightarrow \quad \frac{dy}{1 - y^2} = dx
\]

\[
\int \frac{dy}{1 - y^2} = \int dx
\]

\[
\int \frac{1}{2} \left( \frac{1}{1 - y} + \frac{1}{1 + y} \right) \, dy = \int dx
\]

\[
\frac{1}{2} \ln \left| \frac{1 + y}{1 - y} \right| = x + c
\]

\[
\left| \frac{1 + y}{1 - y} \right| = e^{2x+2c}
\]
After some algebra, we obtain

\[ y = \frac{ke^{2x} - 1}{ke^{2x} + 1} \]

where \( k = e^{2c} \)

**Example 1.6** Solve the initial value problem \( y' = x^3 e^{-y}, \; y(1) = 0 \).

\[
\int e^y \, dy = \int x^3 \, dx
\]

\[ e^y = \frac{x^4}{4} + c \]

\[ y(1) = 0 \; \Rightarrow \; e^0 = \frac{1}{4} + c \]

\[ c = \frac{3}{4} \]

\[ y = \ln \left( \frac{x^4}{4} + \frac{3}{4} \right) \]

### 1.4 Transformations

Sometimes a change of variables simplifies a differential equation just as substitutions simplify integrals. For example if \( y' = f \left( \frac{y}{x} \right) \), the substitution \( u = \frac{y}{x} \) will make the new equation separable.

**Example 1.7** Solve \( y' = \frac{y}{x} + 3 \sqrt{\frac{x}{y}} \).

If \( y = ux \), then \( y' = u'x + u \) and \( u'x + u = u + 3 \sqrt{\frac{1}{u}} \)

\[ u'x = 3 \sqrt{\frac{1}{u}} \]

\[ \sqrt{u} \, du = \frac{3 \, dx}{x} \]
\[
\frac{u^{3/2}}{3/2} = 3 \ln x + c
\]

\[
u = \left(\frac{9}{2} \ln x + c_1\right)^{2/3}
\]

\[
y = x \left(\frac{9}{2} \ln x + c_1\right)^{2/3}
\]

**Example 1.8** Solve the differential equation \((x + y + 6)dx = (-x - y - 3)dy\).

Let’s use the substitution \(u = x + y\). Then,

\[y = u - x, \quad \Rightarrow \quad dy = du - dx\]

and the equation can be expressed in terms of \(u\) and \(x\).

\[(u + 6)dx = (-u - 3)(du - dx)\]

\[3dx = (-u - 3)du\]

\[\int 3dx = \int (-u - 3) du\]

\[3x = -\frac{u^2}{2} - 3u + c\]

\[3x = -\frac{(x + y)^2}{2} - 3(x + y) + c\]

This is an implicit solution.
Exercises

Solve the following differential equations.

1) \( y^3 y' + x^3 = 0 \)
2) \( y' + 4x^3 y^2 = 0 \)
3) \( xy' = x + y \) Hint: \( y' = f \left( \frac{y}{x} \right) \)
4) \( (x^2 + y^2) \, dx + xy \, dy = 0 \) Hint: \( y' = f \left( \frac{y}{x} \right) \)
5) \( y' = xe^{y-x^2} \)
6) \( y' = \frac{1 + \ln x}{4y^3} \)
7) \( y' = 3x^2 \sec^2 y \)
8) \( y' = y(y + 1) \)
9) \( y' + 2y = y^2 + 1 \)
10) \( (1 + y^2) \, dx + x^2 \, dy = 0 \)
11) \( y' = \frac{ay}{x} \)
12) \( y' = e^{ax+by} \)
13) \( y' = x^2 y^2 - 2y^2 + x^2 - 2 \)
14) \( y' = -\frac{2x + y}{x} \)

Solve the following initial value problems:

\[
\begin{align*}
\star 15) & \quad (y^2 + 5xy + 9x^2) \, dx + x^2 \, dy = 0, \quad y(1) = -4 \\
16) & \quad y^3 y' + x^3 = 0, \quad y(0) = 1 \\
17) & \quad y' = -2xy, \quad y(0) = 3 \\
18) & \quad y' = 1 + 4y^2, \quad y(0) = 0 \\
19) & \quad (x^2 + 1)^{1/2} y' = xy^3, \quad y(0) = 2
\end{align*}
\]

\[
\begin{align*}
\star 20) & \quad \frac{dx}{dt} = x \left( \frac{1}{5} - \frac{x^2}{25} \right), \quad x(0) = 1
\end{align*}
\]
Answers

1) $x^4 + y^4 = c$

2) $y = \frac{1}{x^4 + c}$

3) $y = x(\ln|x| + c)$

4) $y^2 = c - \frac{x^2}{2}$

5) $y = -\ln\left(c + \frac{e^{-x^2}}{2}\right)$

6) $y^4 = x \ln x + c$

7) $2y + \sin 2y = 4x^3 + c$

8) $y = \frac{e^x}{c - e^x}$

9) $y = 1 - \frac{1}{x + c}$

10) $y = \tan\left(c + \frac{1}{x}\right)$

11) $y = cx^a$

12) $\frac{e^{ax}}{a} + \frac{e^{-by}}{b} = c$

13) $y = \tan\left(\frac{x^3}{3} - 2x + c\right)$

14) $y = -x + \frac{c}{x}$

15) $y = \frac{x}{\ln x - 1} - 3x$

16) $x^4 + y^4 = 1$

17) $y = 3e^{-x^2}$

18) $y = \frac{1}{2} \tan 2x$

19) $y = \left(\frac{3}{4} - 2\sqrt{x^2 + 1}\right)^{-1/2}$

20) $x = \frac{5e^{t/5}}{4 + e^{3t/5}}$
Chapter 2

Exact and Linear Differential Equations

In this chapter, we will learn how to recognize and solve three different types of equations: Exact, linear, and Bernoulli. All of them are first order equations, therefore we expect a single integration constant in the solution.

At this stage it seems like there’s a special trick for every different kind of question. You will gain familiarity with exercise and experience.

2.1 Exact Equations

A first order differential equation of the form

\[ M(x, y)dx + N(x, y)dy = 0 \] (2.1)

is called an exact differential equation if there exists a function \( u(x, y) \) such that

\[ \frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N \] (2.2)

In other words, \( du = Mdx + Ndy \), so \( Mdx + Ndy \) is a total differential.

For example, the equation

\[ (4x^3 + 2xy^2)dx + (4y^3 + 2x^2y)dy = 0 \]

is exact, and

\[ u = x^4 + x^2y^2 + y^4 \]
Chapter 2. Exact Equations

So, the solution of this equation is very simple, if \( du \) is zero, \( u \) must be a constant, therefore

\[
x^4 + x^2y^2 + y^4 = c
\]

**Theorem 2.1:** The condition \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \) is necessary and sufficient for the equation \( M(x, y)dx + N(x, y)dy = 0 \) to be exact.

**Method of Solution:** To solve \( Mdx + Ndy = 0 \),

- Check for Exactness
- If the equation is exact, find \( u \) by integrating either \( M \) or \( N \).

\[
\begin{align*}
u &= \int Mdx + k(y) \quad \text{or} \quad u = \int Ndy + l(x)
\end{align*}
\]

Note that we have arbitrary functions as integration constants.

- Determine the arbitrary functions using the original equation. The solution is \( u(x, y) = c \)

**Example 2.1** Solve the equation \( 3y^2dx + (3y^2 + 6xy)dy = 0 \).

Let’s check for exactness first.

\[
\frac{\partial(3y^2)}{\partial y} = 6y, \quad \frac{\partial(3y^2 + 6xy)}{\partial x} = 6y
\]

The equation is exact.

\[
u(x, y) = \int 3y^2 dx + k(y) = 3y^2x + k(y)\]

\[
\begin{align*}
\frac{\partial u}{\partial y} &= 6yx + k'(y) = 3y^2 + 6xy \\
k'(y) &= 3y^2 \Rightarrow k(y) = y^3
\end{align*}
\]

We do not need an integration constant here because \( u(x, y) = c \) already contains one

\[
u(x, y) = 3y^2x + y^3 = c
\]
2.2 Integrating Factors

Consider the equation

$$Pdx + Qdy = 0 \quad (2.3)$$

that is not exact. If it becomes exact after multiplying by $F$, i.e. if

$$FPdx + FQdy = 0 \quad (2.4)$$

is exact, then $F$ is called an integrating factor. (Note that $P, Q$ and $F$ are functions of $x$ and $y$)

For example, $ydx - xdy = 0$ is not exact, but $F = \frac{1}{x^2}$ is an integrating factor.

Example 2.2 Solve $(2xe^x - y^2)dx + 2ydy = 0$. Use $F = e^{-x}$.

$$\frac{\partial(2xe^x - y^2)}{\partial y} = -2y, \quad \frac{\partial(2y)}{\partial x} = 0$$

The equation is not exact. Let’s multiply both sides by $e^{-x}$. The new equation is:

$$(2x - y^2e^{-x})dx + 2ye^{-x}dy = 0$$

$$\frac{\partial(2x - y^2e^{-x})}{\partial y} = -2ye^{-x}, \quad \frac{\partial(2ye^{-x})}{\partial x} = -2ye^{-x}$$

Now the equation is exact. We can solve it as we did the previous example and obtain the result

$$x^2 + y^2e^{-x} = c$$

How To Find an Integrating Factor: Let $Pdx + Qdy = 0$ be a differential equation that is not exact, and let $F = F(x, y)$ be an integrating factor. By definition,

$$(FP)_y = (FQ)_x \quad \Rightarrow \quad F_yP + FP_y = F_xQ + FQ_x \quad (2.5)$$
But this equation is more difficult than the one we started with. If we make a simplifying assumption that $F$ is a function of one variable only, we can solve for $F$ and obtain the following theorem:

**Theorem 2.2:** Consider the equation $Pdx + Qdy = 0$. Define

$$R = \frac{1}{Q} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \quad \text{and} \quad \tilde{R} = \frac{1}{P} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \quad (2.6)$$

a) If $R$ depends only on $x$, then $F(x) = \exp(\int R(x) \, dx)$ is an integrating factor.

b) If $\tilde{R}$ depends only on $y$, then $F(y) = \exp(\int \tilde{R}(y) \, dy)$ is an integrating factor.

**Example 2.3** Solve $(4x^2y^2 + 2y)dx + (2x^3y + x)dy = 0$

$$\frac{\partial}{\partial y}(4x^2y^2 + 2y) = 8x^2y + 2, \quad \frac{\partial}{\partial x}(2x^3y + x) = 6x^2y + 1$$

The equation is not exact.

$$R = \frac{8x^2y + 2 - 6x^2y - 1}{2x^4y + x} = \frac{2x^2y + 1}{2x^4y + x} = \frac{1}{x}$$

$$F(x) = e^{\int R(x) \, dx} = e^{\ln x} = x$$

Multiply the equation by $x$ to obtain the exact equation

$$(4x^3y^2 + 2yx)dx + (2x^4y + x^2)dy = 0$$

$$u(x, y) = \int (4x^3y^2 + 2yx) \, dx + k(y) = x^4y^2 + yx^2 + k(y)$$

$$\frac{\partial u}{\partial y} = 2x^4y + x^2 + k'(y) = 2x^4y + x^2 \quad \Rightarrow \quad k(y) = 0$$

$$u(x, y) = x^4y^2 + x^2y = c$$
2.3 Linear First Order Equations

If a first order differential equation can be written in the form

\[ y' + p(x)y = r(x) \]  

(2.7)

it is called a linear differential equation. If \( r(x) = 0 \), the equation is homogeneous, otherwise it is nonhomogeneous.

We can express the equation (2.7) as \( [p(x)y - r(x)]dx + dy = 0 \). This is not exact but it has an integrating factor:

\[ R = p(x), \; F = e^{\int p(x)dx} \]  

(2.8)

**Method of Solution:**

- Given a first order linear equation, express it in the following form:

\[ y' + p(x)y = r(x) \]  

(2.9)

- Multiply both sides by the integrating factor \( F(x) = \exp(\int p(x)dx) \) to obtain

\[ e^{\int p(x)dx}y' + e^{\int p(x)dx}py = re^{\int p(x)dx} \]  

(2.10)

- Express the left hand side as a single parenthesis.

\[ \left( e^{\int p(x)dx}y \right)' = re^{\int p(x)dx} \]  

(2.11)

- Integrate both sides. Don’t forget the integration constant. The solution is:

\[ y(x) = e^{-h} \left( \int e^{h}r dx + c \right) \]  

(2.12)

where \( h = \int p dx \).

**Example 2.4** Solve \( y' + 4y = 1 \)

The integrating factor is \( F = e^{\int 4dx} = e^{4x} \). Multiply both sides of the equation by \( e^{4x} \) to obtain

\[ e^{4x}y' + 4e^{4x}y = e^{4x} \]

\[ (e^{4x}y)' = e^{4x} \]

\[ e^{4x}y = \frac{e^{4x}}{4} + c \Rightarrow \quad y = \frac{1}{4} + ce^{-4x} \]
2.4 Bernoulli Equation

The equation
\[ y' + p(x)y = g(x)y^a \]  
(2.13)
is called Bernoulli equation. It is nonlinear. Nonlinear equations are usually much more difficult than linear ones, but Bernoulli equation is an exception. It can be linearized by the substitution
\[ u(x) = [y(x)]^{1-a} \]  
(2.14)
Then, we can solve it as other linear equations.

Example 2.5 Solve the equation
\[ y' - \frac{2x}{3}y = \frac{e^{x^2}}{3xy^2} \]

Here \( a = -2 \) therefore \( u = y^{1-(-2)} = y^3 \)  \( \Rightarrow \)  \( u' = 3y^2y' \)
Multiplying both sides of the equation by \( 3y^2 \) we obtain
\[ 3y^2y' - 2xy^3 = \frac{e^{x^2}}{x} \Rightarrow u' - 2xu = \frac{e^{x^2}}{x} \]
This equation is linear. Its integrating factor is
\[ e^{\int -2x \, dx} = e^{-x^2} \]
Multiplying both sides by \( e^{-x^2} \), we get
\[ e^{-x^2}y' - 2xe^{-x^2}y' = \frac{1}{x} \]
\[ (e^{-x^2}u)' = \frac{1}{x} \]
\[ e^{-x^2}u = \ln x + c \Rightarrow u = (\ln x + c)e^{x^2} \]
\[ y = \left[ (\ln x + c)e^{x^2} \right]^{1/3} \]
Exercises

Solve the following differential equations. (Find an integrating factor if necessary)

1) $(ye^x + xye^x + 1)dx + xe^x dy = 0$
2) $(2r + 2 \cos \theta) dr - 2r \sin \theta d\theta = 0$
3) $(\sin xy + xy \cos xy) dx + (x^2 \cos xy) dy = 0$
4) $2 \cos ydx = \sin ydy$
5) $5dx - e^{y-x} dy = 0$
6) $(2xy + 3x^2y^6) dx + (4x^2 + 9x^3y^3) dy = 0$
7) $(3xe^y + 2y) dx + (x^2e^y + x) dy = 0$
8) $y' + \frac{5}{x} y = 1$
9) $y' + \frac{1}{x \ln x} y = \frac{1}{\ln x}$
10) $y' - y \tan x = \tan x$
11) $y' + y \tan x = 4x^3 \cos x$
12) $y' + x^3y = 4x^3, y(0) = -1$

Reduce to linear form and solve the following equations:

13) $y' - 4y \tan x = \frac{2 \sin x}{\cos^3 x} y^{1/2}$
14) $y' + y = -\frac{x}{y}$
15) $y' + \frac{25}{x} y = \frac{5 \ln x}{x^5} y^{1/5}$
16) $y' + \frac{y}{x} = -\frac{1}{x^3y^3}$
17) $y = \frac{\tan y}{x - 1}$
18) $y^2 dx + (3xy - 1) dy = 0$

★19) $y'(\sinh 3y - 2xy) = y^2$     Hint: $x \leftrightarrow y$

★20) $2xyy' + (x - 1)y^2 = x^2e^x$     Hint: $z = y^2$
Answers

1) \( y = \left( \frac{c}{x} - 1 \right) e^{-x} \)

2) \( r^2 + 2r \cos \theta = c \)

3) \( x \sin xy = c \)

4) \( F = e^{2x} \), \( e^{2x} \cos y = c \)

5) \( F = e^x \), \( 5e^x - e^y = c \)

6) \( F = y^3 \), \( x^2 y^4 + x^3 y^9 = c \)

7) \( F = x \), \( x^3 e^y + x^2 y = c \)

8) \( y = \frac{1}{5} + \frac{c}{x^5} \)

9) \( y = \frac{x + c}{\ln x} \)

10) \( y = -1 + \frac{c}{\cos x} \)

11) \( y = x^4 \cos x + c \cos x \)

12) \( y = 4 - 5e^{-x^4} \)

13) \( y = \left( \frac{c - \ln \cos x}{\cos^2 x} \right)^2 \)

14) \( y = \sqrt{\frac{1}{2} - x + ce^{-2x}} \)

15) \( y = \left( \frac{x \ln x - x + c}{x^5} \right)^5 \)

16) \( y = \left( \frac{1}{x^8} + \frac{c}{x^4} \right)^{1/4} \)

17) \( y = \arcsin[c(x - 1)] \)

18) \( F = y, x = \frac{1}{2y} + \frac{c}{y^3} \)

19) \( x = y^{-2} \left( \frac{1}{3} \cosh 3y + c \right) \)

20) \( y = \sqrt{cxe^{-x} + \frac{1}{2}xe^x} \)
Chapter 3

Second Order Homogeneous Differential Equations

For first order equations, concepts from calculus and some extensions were sufficient. Now we are starting second order equations and we will learn many new ideas, like reduction of order, linear independence and superposition of solutions.

Many differential equations in applied science and engineering are second order and linear. If in addition they have constant coefficients, we can solve them easily, as explained in this chapter and the next. For nonconstant coefficients, we will have limited success.

3.1 Linear Differential Equations

If we can express a second order differential equation in the form

\[ y'' + p(x)y' + q(x)y = r(x) \]  \hspace{1cm} (3.1)

it is called linear. Otherwise, it is nonlinear.

Consider a linear differential equation. If \( r(x) = 0 \) it is called homogeneous, otherwise it is called nonhomogeneous. Some examples are:

- \( y'' + y^2 = x^2y \) \hspace{1cm} Nonlinear
- \( \sin xy'' + \cos xy = 4 \tan x \) \hspace{1cm} Linear Nonhomogeneous
- \( x^2y'' + y = 0 \) \hspace{1cm} Linear Homogeneous
Chapter 3. Second Order Equations

Linear Combination: A linear combination of \( y_1, y_2 \) is \( y = c_1 y_1 + c_2 y_2 \).

Theorem 3.1: For a homogeneous linear differential equation any linear combination of solutions is again a solution.

The above result does NOT hold for nonhomogeneous equations. For example, both \( y = \sin x \) and \( y = \cos x \) are solutions to \( y'' + y = 0 \), so is \( y = 2 \sin x + 5 \cos x \).

Both \( y = \sin x + x \) and \( y = \cos x + x \) are solutions to \( y'' + y = x \), but \( y = \sin x + \cos x + 2x \) is not.

This is a very important property of linear homogeneous equations, called superposition. It means we can multiply a solution by any number, or add two solutions, and obtain a new solution.

Linear Independence: Two functions \( y_1, y_2 \) are linearly independent if \( c_1 y_1 + c_2 y_2 = 0 \) \( \Rightarrow c_1 = 0, c_2 = 0 \). Otherwise they are linearly dependent. (One is a multiple of the other).

For example, \( e^x \) and \( e^{2x} \) are linearly independent. \( e^x \) and \( 2e^x \) are linearly dependent.

General Solution and Basis: Given a second order, linear, homogeneous differential equation, the general solution is:

\[
y = c_1 y_1 + c_2 y_2
\]

where \( y_1, y_2 \) are linearly independent. The set \( \{y_1, y_2\} \) is called a basis, or a fundamental set of the differential equation.

As an illustration, consider the equation \( x^2 y'' - 5xy' + 8y = 0 \). You can easily check that \( y = x^2 \) is a solution. (We will see how to find it in the last section) Therefore \( 2x^2, 7x^2 \) or \( -x^2 \) are also solutions. But all these are linearly dependent.

We expect a second, linearly independent solution, and this is \( y = x^4 \). A combination of solutions is also a solution, so \( y = x^2 + x^4 \) or \( y = 10x^2 - 5x^4 \) are also solutions. Therefore the general solution is

\[
y = c_1 x^2 + c_2 x^4
\]

and the basis of solutions is \( \{x^2, x^4\} \).
3.2 Reduction of Order

If we know one solution of a second order homogeneous differential equation, we can find the second solution by the method of reduction of order.

Consider the differential equation
\[ y'' + p y' + q y = 0 \]  \hspace{1cm} (3.4)

Suppose one solution \( y_1 \) is known, then set \( y_2 = u y_1 \) and insert in the equation. The result will be
\[ y_1 u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0 \]  \hspace{1cm} (3.5)

But \( y_1 \) is a solution, so the last term is canceled. So we have
\[ y_1 u'' + (2y_1' + py_1)u' = 0 \]  \hspace{1cm} (3.6)

This is still second order, but if we set \( w = u' \), we will obtain a first order equation:
\[ y_1 w' + (2y_1' + py_1)w = 0 \]  \hspace{1cm} (3.7)

Solving this, we can find \( w \), then \( u \) and then \( y_2 \).

**Example 3.1** Given that \( y_1 = x^2 \) is a solution of
\[ x^2 y'' - 3xy' + 4y = 0 \]
find a second linearly independent solution.

Let \( y_2 = ux^2 \). Then
\[ y'_2 = u' x^2 + 2x u \]
and
\[ y''_2 = u'' x^2 + 4x u' + 2u \]
Inserting these in the equation, we obtain
\[ x^4 u'' + x^3 u' = 0 \]
If \( w = u' \) then
\[ x^4 w' + x^3 w = 0 \text{ or } w' + \frac{1}{x} w = 0 \]
This linear first order equation gives \( w = \frac{1}{x} \), therefore \( u = \ln x \) and
\[ y_2 = x^2 \ln x \]
3.3 Homogeneous Equations with Constant Coefficients

Up to now we have studied the theoretical aspects of the solution of linear homogeneous differential equations. Now we will see how to solve the constant coefficient equation $y'' + ay' + by = 0$ in practice.

We have the sum of a function and its derivatives equal to zero, so the derivatives must have the same form as the function. Therefore we expect the function to be $e^{\lambda x}$. If we insert this in the equation, we obtain:

$$\lambda^2 + a\lambda + b = 0 \quad (3.8)$$

This is called the characteristic equation of the homogeneous differential equation $y'' + ay' + by = 0$.

If we solve the characteristic equation, we will see three different possibilities:

- Two real roots, double real root and complex conjugate roots.

**Two Real Roots:** The general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (3.9)$$

**Example 3.2** Solve $y'' - 3y' - 10y = 0$

Try $y = e^{\lambda x}$. The characteristic equation is $\lambda^2 - 3\lambda - 10 = 0$ with solution $\lambda_1 = 5$, $\lambda = -2$, so the general solution is

$$y = c_1 e^{5x} + c_2 e^{-2x}$$

**Example 3.3** Solve the initial value problem $y'' - y = 0$, $y(0) = 2$, $y'(0) = 4$

We start with $y = e^{\lambda x}$ as usual. The characteristic equation is $\lambda^2 - 1 = 0$. Therefore $\lambda = \pm 1$. The general solution is: $y = c_1 e^{x} + c_2 e^{-x}$

Now, we have to use the initial values to determine the constants.

$y(0) = 2 \quad \Rightarrow \quad c_1 + c_2 = 2$ and $y'(0) = 4 \quad \Rightarrow \quad c_1 - c_2 = 4$.

By solving this system, we obtain $c_1 = 3$, $c_2 = -1$ so the particular solution is:

$$y = 3e^{x} - e^{-x}$$
Double Real Root: One solution is $e^{\lambda x}$ but we know that a second order equation must have two independent solutions. Let’s use the method of reduction of order to find the second solution.

$$y'' - 2ay' + a^2 y = 0 \quad \Rightarrow \quad y_1 = e^{ax} \quad (3.10)$$

Let’s insert $y_2 = ue^{ax}$ in the equation.

$$e^{ax}u'' + (2a - 2a)e^{ax}u' = 0 \quad (3.11)$$

Obviously, $u'' = 0$ therefore $u = c_1 + c_2 x$. The general solution is

$$y = c_1 e^{\lambda x} + c_2 xe^{\lambda x} \quad (3.12)$$

Example 3.4 Solve $y'' + 2y' + y = 0$

$y = e^{\lambda x}$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$. Its solution is the double root $\lambda = -1$, therefore the general solution is

$$y = c_1 e^{-x} + c_2 xe^{-x}$$

Complex Conjugate Roots: We need the complex exponentials for this case. Euler’s formula is

$$e^{ix} = \cos x + i \sin x \quad (3.13)$$

This can be proved using Taylor series expansions.

If the solution of the characteristic equation is

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta \quad (3.14)$$

then the general solution of the differential equation will be

$$y = c_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + c_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) \quad (3.15)$$

By choosing new constants $A, B$, we can express this as

$$y = e^{\alpha x} (A \cos \beta x + B \sin \beta x) \quad (3.16)$$

Example 3.5 Solve $y'' - 4y' + 29y = 0$.

$y = e^{\lambda x}$. The characteristic equation is $\lambda^2 - 4\lambda + 29 = 0$. Therefore $\lambda = 2 \pm 5i$.

The general solution is

$$y = e^{2x} (A \cos 5x + B \sin 5x)$$
3.4 Cauchy-Euler Equation

The equation \( x^2 y'' + ax y' + by = 0 \) is called the Cauchy-Euler equation. By inspection, we can easily see that the solution must be a power of \( x \). Let’s substitute \( y = x^r \) in the equation and try to determine \( r \). We will obtain

\[
\begin{align*}
r(r-1)x^r + ax^r + bx^r &= 0 \\
r^2 + (a-1)r + b &= 0
\end{align*}
\]

This is called the auxiliary equation. Once again, we have three different cases according to the types of roots. The general solution is given as follows:

- **Two real roots**

  \[ y = c_1 x^{r_1} + c_2 x^{r_2} \]  \hspace{1cm} (3.19)

- **Double real root**

  \[ y = c_1 x^r + c_2 x^r \ln x \]  \hspace{1cm} (3.20)

- **Complex conjugate roots** where \( r_1, r_2 = r \pm si \)

  \[ y = x^r \left[ c_1 \cos(s \ln x) + c_2 \sin(s \ln x) \right] \]  \hspace{1cm} (3.21)

**Example 3.6** Solve \( x^2 y'' + 2xy' - 6y = 0 \)

Insert \( y = x^r \). Auxiliary equation is \( r^2 + r - 6 = 0 \). The roots are \( r = 2, r = -3 \) therefore

\[ y = c_1 x^2 + c_2 x^{-3} \]

**Example 3.7** Solve \( x^2 y'' - 9xy' + 25y = 0 \)

Insert \( y = x^r \). Auxiliary equation is \( r^2 - 10r + 25 = 0 \). Auxiliary equation has the double root \( r = 5 \) therefore the general solution is

\[ y = c_1 x^5 + c_2 x^5 \ln x \]
Exercises

Are the following sets linearly independent?
1) \{x^4, x^8\}
2) \{\sin x, \sin^2 x\}
3) \{\ln(x^5), \ln x\}

Use reduction of order to find a second linearly independent solution:
★ 4) \(x^2(\ln x - 1)y'' - xy' + y = 0, \quad y_1 = x\)
5) \(x^2 \ln x y'' + (2x \ln x - x)y' - y = 0, \quad y_1 = \frac{1}{x}\)
6) \(y'' + 3 \tan x y' + (3 \tan^2 x + 1)y = 0, \quad y_1 = \cos x\)

Solve the following equations:
7) \(y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0\)
8) \(y'' + \frac{5}{2}y' + y = 0\)
9) \(y'' - 64y = 0, \quad y(0) = 1, \quad y'(0) = 8\)
10) \(y'' + 24y' + 144y = 0\)
11) \(y'' + 2y' + y = 0, \quad y(-1) = e, \quad y(1) = \frac{7}{e}\)
12) \(5y'' - 8y' + 5y = 0\)
13) \(y'' + 2y' + \left(1 + \frac{\pi^2}{4}\right)y = 0, \quad y(0) = 1, \quad y'(0) = -1\)
14) \(y'' - 2y' + 2y = 0, \quad y(\pi) = 0, \quad y(-\pi) = 0\)
15) \(xy'' + y' = 0\)
16) \(x^2y'' - 3xy' + 5y = 0\)
17) \(x^2y'' - 10xy' + 18y = 0\)
18) \(x^2y'' - 13xy' + 49y = 0\)
19) Show that \(y_1 = u\) and \(y_2 = u \int v dx\) are solutions of the equation
\[y'' - \left(\frac{v'}{v} + 2 \frac{u'}{u}\right)y' + \left(\frac{v' u'}{vu} + 2 \frac{u'^2}{u^2} - \frac{u''}{u}\right)y = 0\]
20) Show that \(y_1 = u\) and \(y_2 = v\) are solutions of the equation
\[(uv'/vu')y'' + (vu'' - uv'')y' + (u'v'' - v'u'')y = 0\]
Answers
1) Yes
2) Yes
3) No
4) $y_2 = \ln x$
5) $y_2 = \ln x - 1$

6) $y_2 = \sin x \cos x$

7) $y = (1 + x)e^{-x}$

8) $y = c_1 e^{-2x} + c_2 e^{-12x}$

9) $y = e^{8x}$

10) $y = c_1 e^{-12x} + c_2 xe^{-12x}$

11) $y = 4e^{-x} + 3xe^{-x}$

12) $y = e^{0.8x} [A \cos(0.6x) + B \sin(0.6x)]$

13) $y = e^{-x} \cos \left( \frac{\pi}{2} x \right)$

14) $y = e^x \sin x$

15) $y = c_1 + c_2 \ln x$

16) $y = x^2 [c_1 \cos(\ln x) + c_2 \sin(\ln x)]$

17) $y = c_1 x^2 + c_2 x^9$

18) $y = c_1 x^7 + c_2 x^7 \ln x$
Chapter 4

Second Order Nonhomogeneous Equations

In this chapter we will start to solve the nonhomogeneous equations, and see that we will need the homogeneous solutions we found in the previous chapter.

Of the two methods we will learn, undetermined coefficients is simpler, but it can be applied to a restricted class of problems. Variation of parameters is more general but involves more calculations.

4.1 General and Particular Solutions

Consider the nonhomogeneous equation
\[ y'' + p(x)y' + q(x)y = r(x) \]  
(4.1)

Let \( y_p \) be a solution of this equation. Now consider the corresponding homogeneous equation
\[ y'' + p(x)y' + q(x)y = 0 \]  
(4.2)

Let \( y_h \) be the general solution of this one. If we add \( y_h \) and \( y_p \), the result will still be a solution for the nonhomogeneous equation, and it must be the general solution because \( y_h \) contains two arbitrary constants. This interesting property means that we need the homogeneous equation when we are solving
CHAPTER 4. NONHOMOGENEOUS EQUATIONS

the nonhomogeneous one. The general solution is of the form

\[ y = y_h + y_p \]  \hspace{1cm} (4.3)

**Example 4.1** Find the general solution of \( y'' - 3y' + 2y = 2x - 3 \) using \( y_p = x \).

Let's solve \( y'' - 3y' + 2y = 0 \) first. Let \( y_h = e^{\lambda x} \). Then

\[ \lambda^2 - 3\lambda + 2 = 0 \]

which means \( \lambda = 2 \) or \( \lambda = 1 \). The homogenous solution is

\[ y_h = c_1e^x + c_2e^{2x} \]

therefore the general solution is:

\[ y = x + c_1e^x + c_2e^{2x} \]

**Example 4.2** Find the general solution of \( y'' = \cos x \) using \( y_p = -\cos x \).

The solution of \( y'' = 0 \) is simply \( y_h = c_1 + c_2 \), therefore the general solution must be

\[ y = -\cos x + c_1x + c_2 \]

As you can see, once we have a particular solution, the rest is straightforward, but how can we find \( y_p \) for a given equation?

**Example 4.3** Find a particular solution of the following differential equations. Try the suggested functions. (Success not guaranteed!)

a) \( y'' + y = e^x \), \hspace{1cm} Try \( y_p = Ae^x \)
b) \( y'' - y = e^x \), \hspace{1cm} Try \( y_p = Ae^x \)
c) \( y'' + 2y' + y = x \), \hspace{1cm} Try \( y_p = Ax + B \)
d) \( y'' + 2y' = x \), \hspace{1cm} Try \( y_p = Ax + B \)
e) \( y'' + 2y' + y = 2\cos x \), \hspace{1cm} Try \( y_p = A\cos x \) and \( y_p = A\cos x + B\sin x \)

As you can see, some of the suggestions work and some do not.

\( y_p \) is usually similar to \( r(x) \). We can summarize our findings as:

- Start with a set of functions that contains not only \( r(x) \), but also all derivatives of \( r(x) \).
- If one of the terms of \( y_p \) candidate occurs in \( y_h \), there is a problem.
4.2. Method of Undetermined Coefficients

To solve the constant coefficient equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = r(x)$$  \hspace{1cm} (4.4)

- Solve the corresponding homogeneous equation, find $y_h$.
- Find a candidate for $y_p$ using the following table:

<table>
<thead>
<tr>
<th>Term in $r(x)$</th>
<th>Choice for $y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$A_n x^n + \cdots + A_1 x + A_0$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$A e^{ax}$</td>
</tr>
<tr>
<td>$\cos bx$ or $\sin bx$</td>
<td>$A \cos bx + B \sin bx$</td>
</tr>
<tr>
<td>$x^n e^{ax}$</td>
<td>$(A_n x^n + \cdots + A_1 x + A_0) e^{ax}$</td>
</tr>
<tr>
<td>$x^n \cos bx$ or $x^n \sin bx$</td>
<td>$(A_n x^n + \cdots + A_0) \cos bx + (B_n x^n + \cdots + B_0) \sin bx$</td>
</tr>
<tr>
<td>$e^{ax} \cos bx$ or $e^{ax} \sin bx$</td>
<td>$A e^{ax} \cos bx + B e^{ax} \sin bx$</td>
</tr>
<tr>
<td>$x^n e^{ax} \cos bx$ or $x^n e^{ax} \sin bx$</td>
<td>$(A_n x^n + \cdots + A_0) e^{ax} \cos bx + (B_n x^n + \cdots + B_0) e^{ax} \sin bx$</td>
</tr>
</tbody>
</table>

(You don’t have to memorize the table. Just note that the choice consists of $r(x)$ and all its derivatives)

- If your choice for $y_p$ occurs in $y_h$, you have to change it. Multiply it by $x$ if the solution corresponds to a single root, by $x^2$ if it is a double root.

- Find the constants in $y_p$ by inserting it in the equation.

- The general solution is $y = y_p + y_h$

Note that this method works only for constant coefficient equations, and only when $r(x)$ is relatively simple.

**Example 4.4** Find the general solution of the equation

$$3y'' + 10y' + 3y = 9x$$
The homogeneous equation is

\[ 3y'' + 10y' + 3y = 0 \]

Its solution is

\[ y_h = c_1 e^{-3x} + c_2 e^{-x/3} \]

To find a particular solution, let’s try \( y_p = Ax + B \). Inserting this in the equation, we obtain:

\[ 10A + 3A + 3B = 9 \]

Therefore, \( A = 3 \), \( B = -10 \). The particular solution is:

\[ y_p = 3x - 10 \]

The general solution is:

\[ y = c_1 e^{-3x} + c_2 e^{-x/3} + 3x - 10 \]

**Example 4.5** *Find the general solution of* \( y'' - 4y' + 4y = e^{2x} \)

The solution of the associated homogeneous equation

\[ y'' - 4y' + 4y = 0 \]

is

\[ y_h = c_1 e^{2x} + c_2 xe^{2x} \]

Our candidate for \( y_p \) is \( y_p = Ae^{2x} \). But this is already in the \( y_h \) so we have to change it. If we multiply by \( x \), we will obtain \( Axe^{2x} \) but this is also in \( y_h \). Therefore we have to multiply by \( x^2 \). So our choice for \( y_p \) is \( y_p = Ax^2 e^{2x} \). Now we have to determine \( A \) by inserting in the equation.

\[ y_p' = 2Ax^2 e^{2x} + 2Axe^{2x} \]

\[ y_p'' = 4Ax^2 e^{2x} + 8Axe^{2x} + 2Ae^{2x} \]

\[ 4Ax^2 e^{2x} + 8Axe^{2x} + 2Ae^{2x} - 4(2Ax^2 e^{2x} + 2Axe^{2x}) + 4Ax^2 e^{2x} = e^{2x} \]
4.3 Method of Variation of Parameters

Consider the linear second order nonhomogeneous differential equation

\[ a(x)y'' + b(x)y' + c(x)y = r(x) \]  \hspace{1cm} (4.5)

If \( a(x), b(x) \) and \( c(x) \) are not constants, or if \( r(x) \) is not among the functions given in the table, we can not use the method of undetermined coefficients. In this case, the variation of parameters can be used if we know the homogeneous solution.

Let \( y_h = c_1y_1 + c_2y_2 \) be the solution of the associated homogeneous equation

\[ a(x)y'' + b(x)y' + c(x)y = 0 \]  \hspace{1cm} (4.6)

Let us express the particular solution as:

\[ y_p = v_1(x)y_1 + v_2(x)y_2 \]  \hspace{1cm} (4.7)

There are two unknowns, so we may impose an extra condition. Let’s choose \( v'_1y_1 + v'_2y_2 = 0 \) for simplicity. Inserting \( y_p \) in the equation, we obtain

\[ v'_1y'_1 + v'_2y'_2 = \frac{r}{a} \]
\[ v'_1y_1 + v'_2y_2 = 0 \]  \hspace{1cm} (4.8)

The solution to this linear system is

\[ v'_1 = \frac{-y_2r}{aW}, \quad v'_2 = \frac{y_1r}{aW} \]  \hspace{1cm} (4.9)

where \( W \) is the Wronskian

\[ W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1 \]  \hspace{1cm} (4.10)
Therefore the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 r}{aW} \, dx + y_2 \int \frac{y_1 r}{aW} \, dx$$

(4.11)

**Example 4.6** Find the general solution of $y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$

$$y_h = c_1 e^{-x} + c_2 x e^{-x}$$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

$$y_p = -e^{-x} \int \frac{xe^{-x} e^{-x}}{e^{-2x} \sqrt{x}} \, dx + xe^{-x} \int \frac{e^{-x} e^{-x}}{e^{-2x} \sqrt{x}} \, dx$$

$$y_p = -e^{-x} \int \sqrt{x} \, dx + xe^{-x} \int \frac{1}{\sqrt{x}} \, dx$$

$$y_p = -e^{-x} \frac{x^{3/2}}{3/2} + xe^{-x} \frac{x^{1/2}}{1/2} = \frac{4}{3} e^{-x} x^{3/2}$$

$$y = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + \frac{4}{3} e^{-x} x^{3/2}$$

**Example 4.7** Find the general solution of $x^2 y'' - 5xy' + 8y = x^5$

We can find the homogeneous solution of the Cauchy-Euler equation as:

$$y_h = c_1 x^4 + c_2 x^2$$

$$W = \begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix} = -2x^5$$

Therefore the particular solution is

$$y_p(x) = -x^4 \int \frac{x^2 x^5}{x^2 (-2x^5)} \, dx + x^2 \int \frac{x^4 x^5}{x^2 (-2x^5)} \, dx$$

$$y_p(x) = \frac{1}{2} x^4 \int dx - \frac{1}{2} x^2 \int x^2 \, dx$$

$$y_p(x) = \frac{1}{3} x^5$$

The general solution is

$$y = c_1 x^4 + c_2 x^2 + \frac{1}{3} x^5$$
Exercises

Find the general solution of the following differential equations

1) \[ y'' + 4y = x \cos x \]
2) \[ y'' - 18y' + 81y = e^{9x} \]
3) \[ y'' = -4x \cos 2x - 4 \cos 2x - 8x \sin 2x - 8 \sin 2x \]
4) \[ y'' + 3y' - 18y = 9 \sinh 3x \]
5) \[ y'' + 16y = x^2 + 2x \]
6) \[ y'' - 2y' + y = x^2 e^x \]
7) \[ 2x^2 y'' - xy' + y = \frac{1}{x} \]

8) \[ x^2 y'' + xy' - 4y = x^2 \ln x \]
9) \[ y'' - 8y' + 16y = 16x \]
10) \[ y'' = x^3 \]
11) \[ y'' + 7y' + 12y = e^{2x} + x \]
12) \[ y'' + 12y' + 36y = 100 \cos 2x \]

13) \[ y'' + 9y = e^x + \cos 3x + 2 \sin 3x \]
14) \[ y'' + 10y' + 16y = e^{-2x} \]
15) \[ y'' - 4y' + 53y = (53x)^2 \]
16) \[ y'' + y = (x^2 + 1)e^{3x} \]
17) \[ y'' + y = \csc x \]
18) \[ y'' + y = \csc x \sec x \]
19) \[ y'' - 4y' + 4y = \frac{e^{2x} \ln x}{x} \]

20) \[ y'' - 2y' + y = \frac{e^{2x}}{(e^x + 1)^2} \]
CHAPTER 4. NONHOMOGENEOUS EQUATIONS

Answers

1) \( y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{3} x \cos x + \frac{2}{5} \sin x \)

2) \( y = c_1 e^{9x} + c_2 xe^{9x} + \frac{1}{2} x^2 e^{9x} \)

3) \( y = c_1 + c_2 x + x \cos 2x + 3 \cos 2x + 2x \sin 2x + \sin 2x \)

4) \( y = c_1 e^{3x} + c_2 e^{-6x} + \frac{1}{4} e^{-3x} + \frac{1}{2} xe^{3x} \)

5) \( y = c_1 \sin 4x + c_2 \cos 4x + \frac{1}{16} x^2 + \frac{1}{8} x - \frac{1}{128} \)

6) \( y = c_1 e^x + c_2 xe^x + \frac{1}{12} x^4 e^x \)

7) \( y = c_1 x + c_2 \sqrt{x} + \frac{1}{6} x \)

8) \( y = c_1 x^2 + c_2 x^{-2} + \frac{1}{8} x^2 \ln^2 x - \frac{1}{16} x^2 \ln x + \frac{x^2}{64} \)

9) \( y = c_1 e^{4x} + c_2 xe^{4x} + x + \frac{1}{2} \)

10) \( y = \frac{x^5}{20} + c_1 + c_2 x \)

11) \( y = c_1 e^{-3x} + c_2 e^{-4x} + \frac{1}{30} e^{2x} + \frac{1}{12} x - \frac{7}{144} \)

12) \( y = c_1 e^{-6x} + c_2 xe^{-6x} + 2 \cos 2x + \frac{3}{2} \sin 2x \)

13) \( y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{10} e^x - \frac{1}{3} x \cos 3x + \frac{1}{6} x \sin 3x \)

14) \( y = c_1 e^{-2x} + c_2 e^{-8x} + \frac{1}{6} xe^{-2x} \)

15) \( y = e^{2x} (c_1 \cos 7x + c_2 \sin 7x) + 53x^2 + 8x - \frac{74}{53} \)

16) \( y = e^{3x} (0.1x^2 - 0.12x + 0.152) + c_1 \sin x + c_2 \cos x \)

17) \( y = c_1 \sin x + c_2 \cos x - x \cos x + \sin x \ln x \cos x \)

18) \( y = c_1 \sin x + c_2 \cos x - \cos x \ln x \sec x + \tan x - \sin x \ln x \csc x + \cot x \)

19) \( y = c_1 e^{2x} + c_2 xe^{2x} + xe^{2x} \left[ \frac{(\ln x)^2}{2} - \ln x + 1 \right] \)

20) \( y = c_1 e^x + c_2 xe^x + e^x \ln (1 + e^x) \)
Chapter 5

Higher Order Equations

In this chapter, we will generalize our results about second order equations to higher orders. The basic ideas are the same. We still need the homogeneous solution to find the general nonhomogeneous solution. We will extend the two methods, undetermined coefficients and variation of parameters, to higher dimensions and this will naturally involve many more terms and constants in the solution. We also need some new notation to express \( n \)th derivatives easily.

5.1 Linear Equations of Order \( n \)

An \( n \)th order differential equation is called linear if it can be written in the form

\[
a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x) \quad (5.1)
\]

and nonlinear if it is not linear. (Note that \( a_0 \neq 0 \))

If the coefficients \( a_0(x), a_1(x), \ldots a_n(x) \) are continuous, then the equation has exactly \( n \) linearly independent solutions. The general solution is

\[
y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \quad (5.2)
\]

Linear Independence: If

\[
c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0 \quad (5.3)
\]
means that all the constants \( c_1, c_2, \ldots, c_n \) are zero, then this set of functions is linearly independent. Otherwise, they are dependent.

For example, the functions \( x, x^2, x^3 \) are linearly independent. The functions \( \cos^2 x, \sin^2 x, \cos 2x \) are not.

Given \( n \) functions, we can check their linear dependence by calculating the Wronskian. The Wronskian is defined as

\[
W(y_1, y_2, \ldots, y_n) = \begin{vmatrix}
  y_1 & \cdots & y_n \\
  y'_1 & \cdots & y'_n \\
  \vdots & & \vdots \\
  y^{(n-1)}_1 & \cdots & y^{(n-1)}_n \\
\end{vmatrix}
\]

and the functions are linearly dependent if and only if \( W = 0 \) at some point.

### 5.2 Differential Operators

We can denote differentiation with respect to \( x \) by the symbol \( D \)

\[
Dy = \frac{dy}{dx} = y', \quad D^2y = \frac{d^2y}{dx^2} = y''
\]

etc. A differential operator is

\[
L = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n
\]

We will only work with operators where coefficients are constant.

We can add, multiply, expand and factor constant coefficient differential operators using common rules of algebra. In this respect, they are like polynomials. So, the following expressions are all equivalent:

\[
(D - 2)(D - 3)y = (D - 3)(D - 2)y \\
= (D^2 - 5D + 6)y \\
= y'' - 5y' + 6y
\]

Let’s apply some simple operators to selected functions:

\[
(D - 2)e^x = De^x - 2e^x \\
= e^x - 2e^x = -e^x
\]
5.3. **HOMOGENEOUS EQUATIONS**

\[(D - 2)e^{2x} = De^{2x} - 2e^{2x}\]
\[= 2e^{2x} - 2e^{2x} = 0\]

\[(D - 2)^2xe^{2x} = (D - 2)(D - 2)xe^{2x}\]
\[= (D - 2)(e^{2x} + 2xe^{2x} - 2xe^{2x})\]
\[= (D - 2)e^{2x} = 0\]

\[(D^2 - 4)\sin(2x) = (D - 2)(D + 2)\sin(2x)\]
\[= (D - 2)(2\cos(2x) + 2\sin(2x))\]
\[= -4\sin(2x) + 4\cos(2x) - 4\cos(2x) - 4\sin(2x)\]
\[= -8\sin(2x)\]

5.3 **Homogeneous Equations**

Based on the examples in the previous section, we can easily see that:

The general solution of the equation \((D - a)^ny = 0\) is

\[y = e^{ax}(c_0 + c_1x + \ldots + c_{n-1}x^{n-1})\]  \hspace{1cm} (5.7)

if \(a\) is real.

Some special cases are:

\[D^n y = 0 \Rightarrow y = c_0 + c_1x + \ldots + c_{n-1}x^{n-1}\]
\[(D - a)y = 0 \Rightarrow y = e^{ax}\]  \hspace{1cm} (5.8)
\[(D - a)^2y = 0 \Rightarrow y = c_1e^{ax} + c_2xe^{ax}\]

We can extend these results to the case of complex roots. If \(z = a + ib\) is a root of the characteristic polynomial, then so is \(z = a - ib\). (Why?)

Consider the equation

\[(D - a - ib)(D - a + ib)^ny = (D^2 - 2aD + a^2 + b^2)^ny = 0\]  \hspace{1cm} (5.9)

The solution is

\[y = e^{ax}\cos bx(c_0 + c_1x + \ldots + c_{n-1}x^{n-1})\]
\[+ e^{ax}\sin bx(k_0 + k_1x + \ldots + k_{n-1}x^{n-1})\]  \hspace{1cm} (5.10)

A special case is obtained if \(a = 0\).

\[(D^2 + b^2)y = 0 \Rightarrow y = c_1\cos bx + c_2\sin bx\]  \hspace{1cm} (5.11)
Now we are in a position to solve very complicated-looking homogeneous equations.

**Method of Solution:**

- Express the given equation using operator notation \((D)\) notation.
- Factor the polynomial.
- Find the solution for each component.
- Add the components to obtain the general solution.

**Example 5.1** Find the general solution of \(y^{(4)} - 7y''' + y'' - 7y' = 0\).

In operator notation, we have

\[
(D^4 - 7D^3 + D^2 - 7D)y = 0
\]

Factoring this, we obtain

\[
D(D - 7)(D^2 + 1)y = 0
\]

We know that

\[
\begin{align*}
Dy &= 0 \quad \Rightarrow \quad y = c \\
(D - 7)y &= 0 \quad \Rightarrow \quad y = ce^{7x} \\
(D^2 + 1)y &= 0 \quad \Rightarrow \quad y = c_1 \sin x + c_2 \cos x
\end{align*}
\]

Therefore the general solution is

\[
y = c_1 + c_2 e^{7x} + c_3 \sin x + c_4 \cos x
\]

Note that the equation is fourth order and the solution has four arbitrary constants.

**Example 5.2** Solve \(D^3(D - 2)(D - 3)^2(D^2 + 4)y = 0\).

Using the same method, we find:

\[
y = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^{3x} + c_6 e^{3x} x + c_7 \cos 2x + c_8 \sin 2x
\]
5.4 Nonhomogeneous Equations

In this section, we will generalize the methods of undetermined coefficients and variation of parameters to \( n \)th order equations.

**Undetermined Coefficients:** Method of undetermined coefficients is the same as given on page 27. We will use the same table, but this time the modification rule is more general. It should be:

- In case one of the terms of \( y_p \) occurs in \( y_h \), multiply it by \( x^k \) where \( k \) is the smallest integer which will eliminate any duplication between \( y_p \) and \( y_h \).

**Example 5.3** Solve the equation \((D - 1)^4 y = xe^x\).

The homogeneous solution is \( y_h = (c_0 + c_1 x + c_2 x^2 + c_3 x^3) e^x \). According to the table, we should choose \( y_p \) as \( Ae^x + Bxe^x \), but this already occurs in the homogeneous solution. Multiplying by \( x, x^2, x^3 \) are not enough, so, we should multiply by \( x^4 \).

\[ y_p = Ax^4e^x + Bx^5e^x \]

Inserting this in the equation, we obtain:

\[ 24Ae^x + 120Bxe^x = xe^x \]

Therefore \( A = 0, B = 1/120 \) and the general solution is

\[ y = (c_0 + c_1 x + c_2 x^2 + c_3 x^3) e^x + \frac{1}{120} x^5 e^x \]

**Variation of Parameters:** The idea is the same as in second order equations, but there are more unknowns to find and more integrals to evaluate. Consider

\[ a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = r(x) \quad (5.12) \]

Let the homogeneous solution be \( y_h = c_1 y_1 + \cdots + c_n y_n \)

Then the particular solution is \( y_p = v_1 y_1 + \cdots + v_n y_n \)

Here, \( v_i \) are functions of \( x \). Since we have \( n \) functions, we can impose \( n - 1 \) conditions on them. The first condition will be

\[ v_1' y_1 + \cdots + v_n' y_n = 0 \quad (5.13) \]
Then we will proceed similarly to simplify the steps. Eventually, we will obtain the system

\[
\begin{align*}
 v_1'y_1 + \cdots + v_n'y_n &= 0 \\
 v_1'y_1' + \cdots + v_n'y_n' &= 0 \\
 & \vdots \\
 v_1'y_1^{(n-1)} + \cdots + v_n'y_n^{(n-1)} &= 0 \\
 v_1'y_1^{(n)} + \cdots + v_n'y_n^{(n)} &= \frac{r(x)}{a_0(x)}
\end{align*}
\]  

(5.14)

Then, we will solve this linear system to find \( v'_i \), and integrate them to obtain \( y_p \).

\[
y_p = y_1 \int v_1' \, dx + \cdots + y_n \int v_n' \, dx
\]

(5.15)

**Example 5.4** Find the general solution of

\[
x^3 y''' - 6x^2 y'' + 15xy' - 15y = 8x^6
\]

We can find the homogeneous solution \( y_h = c_1x + c_2x^3 + c_3x^5 \) using our method for Cauchy-Euler equations. Then, the particular solution will be \( y_p = xv_1 + x^3v_2 + x^5v_3 \). Using the above equations, we obtain the system

\[
\begin{align*}
 xv_1' + x^3v_2' + x^5v_3' &= 0 \\
 v_1' + 3x^2v_2' + 5x^4v_3' &= 0 \\
 6xv_2' + 20x^3v_3' &= 8x^3
\end{align*}
\]

The solution of this system is \( v_1' = x^4, \ v_2' = -2x^2, \ v_3' = 1 \) therefore the particular solution is

\[
y_p = x \int x^4 \, dx + x^3 \int (-2x^2) \, dx + x^5 \int \, dx = \frac{8}{15}x^6
\]

and the general solution is

\[
y = c_1x + c_2x^3 + c_3x^5 + \frac{8}{15}x^6
\]
Exercises

1) \( D^5y = 0 \)
2) \((D - 1)^3y = 0 \)
3) \( y''' - 4y'' + 13y' = 0 \)
4) \((D - 2)^2(D + 3)^3y = 0 \)
5) \((D^2 + 2)^3y = 0 \)
6) \( \frac{d^4y}{dx^4} + \frac{5d^2y}{dx^2} + 4y = 0 \)
7) \((D^2 + 9)^2(D^2 - 9)^2y = 0 \)
8) \( \frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0 \)
9) \( y''' - 3y'' + 12y' - 10y = 0 \)
10) \((D^2 + 2D + 17)^2y = 0 \)
11) \((D^4 + 2D^2 + 1)y = x^2 \)
12) \((D^3 + 2D^2 - D - 2)y = 1 - 4x^3 \)
13) \( (2D^4 + 4D^3 + 8D^2)y = 40e^{-x}[\sqrt{3} \sin(\sqrt{3}x) + 3 \cos(\sqrt{3}x)] \)
14) \( (D^3 - 4D^2 + 5D - 2)y = 4 \cos x + \sin x \)
15) \( (D^3 - 9D)y = 8xe^x \)
Answers
1) \( y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \)
2) \( y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x \)
3) \( y = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x + c_3 \)
4) \( y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x} + c_4 x e^{-3x} + c_5 x^2 e^{-3x} \)
5) \( y = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x + c_3 x \cos \sqrt{2} x + c_4 x \sin \sqrt{2} x \\
\quad + c_5 x^2 \cos \sqrt{2} x + c_6 x^2 \sin \sqrt{2} x \)
6) \( y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x \)
7) \( y = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-3x} + c_4 x e^{-3x} + c_5 \cos 3x + c_6 \sin 3x \\
\quad + c_7 x \cos 3x + c_8 x \sin 3x \)
8) \( y = c_1 + c_2 x + c_3 e^x \cos x + c_4 e^x \sin x \)
9) \( y = c_1 e^x + c_2 e^x \cos 3x + c_3 e^x \sin 3x \)
10) \( y = c_1 e^{-x} \sin 4x + c_2 e^{-x} \cos 4x + c_3 x e^{-x} \sin 4x + c_4 x e^{-x} \cos 4x \)
11) \( y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + x^2 - 4 \)
12) \( y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} + 2x^3 - 3x^2 + 15x - 8 \)
13) \( y = c_1 + c_2 x + c_3 e^{-x} \cos \sqrt{3} x + c_4 e^{-x} \sin \sqrt{3} x + 5 xe^{-x} \cos \sqrt{3} x \)
14) \( y = c_1 e^x + c_2 x e^x + c_3 e^{2x} + 0.2 \cos x + 0.9 \sin x \)
15) \( y = c_1 + c_2 e^{3x} + c_3 e^{-3x} + \frac{3}{4} e^x - xe^x \)
Chapter 6

Series Solutions

If none of the methods we have studied up to now works for a differential equation, we may use power series. This is usually the only choice if the solution cannot be expressed in terms of the elementary functions. (That is, exponential, logarithmic, trigonometric and polynomial functions). If the solution can be expressed as a power series, in other words, if it is analytic, this method will work. But it takes time and patience to reach the solution. Remember, we are dealing with infinitely many coefficients!

6.1 Power Series

Let’s remember some facts about the series

\[ \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \]  

(6.1)

from calculus.

- There is a nonnegative number \( \rho \), called the radius of convergence, such that the series converges absolutely for \( |x - x_0| < \rho \) and diverges for \( |x - x_0| > \rho \). The series defines a function \( f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \) in its interval of convergence.

- In the interval of convergence, the series can be added or subtracted
term wise, i.e.

\[ f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n \]

- In the interval of convergence, the series can be multiplied or divided to give another power series.

\[ f(x)g(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n \]

where

\[ c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0 \]

- In the interval of convergence, derivatives and integrals of \( f(x) \) can be found by term wise differentiation and integration, for example

\[ f'(x) = a_1 + 2a_2(x - x_0) + \cdots = \sum_{n=1}^{\infty} n a_n(x - x_0)^{n-1} \]

- The series \( \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \) is called the Taylor Series of the function \( f(x) \). The function \( f(x) \) is called **analytic** if its Taylor series converges.

Examples of some common power series are:

\[
\begin{align*}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots
\end{align*}
\]
6.2 Classification of Points

Consider the equation
\[ R(x)y'' + P(x)y' + Q(x)y = 0 \]  (6.2)

If both of the functions
\[ \frac{P(x)}{R(x)}, \quad \frac{Q(x)}{R(x)} \]  (6.3)

are analytic at \( x = x_0 \), then the point \( x_0 \) is an ordinary point. Otherwise, \( x_0 \) is a singular point.

Suppose that \( x_0 \) is a singular point of the above equation. If both of the functions
\[ (x - x_0)\frac{P(x)}{R(x)}, \quad (x - x_0)^2\frac{Q(x)}{R(x)} \]  (6.4)

are analytic at \( x = x_0 \), then the point \( x_0 \) is called a regular singular point. Otherwise, \( x_0 \) is an irregular singular point.

For example, the functions \( 1 + x + x^2, \sin x, e^x(1 + x^4) \cos x \) are all analytic at \( x = 0 \). But, the functions \( \frac{\cos x}{x}, \frac{1}{x}, \frac{e^x}{x}, \frac{1 + x^2}{x^3} \) are not.

We will use power series method around ordinary points and Frobenius’ method around regular singular points. We will not consider irregular singular points.

6.3 Power Series Method

If \( x_0 \) is an ordinary point of the equation \( R(x)y'' + P(x)y' + Q(x)y = 0 \), then the general solution is
\[ y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \]  (6.5)

The coefficients \( a_n \) can be found by inserting \( y \) in the equation and setting the coefficients of all powers to zero. Two coefficients (Usually \( a_0 \) and \( a_1 \)) must be arbitrary, others must be defined in terms of them. We expect two linearly independent solutions because the equation is second order linear.
Example 6.1 Solve $y'' + 2xy' + 2y = 0$ around $x_0 = 0$.

First we should classify the point. Obviously, $x = 0$ is an ordinary point, so we can use power series method.

$$ y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} $$

Inserting these in the equation, we obtain

$$ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x \sum_{n=1}^{\infty} na_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0 $$

To equate the powers of $x$, let us replace $n$ by $n + 2$ in the first sigma. ($n \to n + 2$)

$$ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 $$

Now we can express the equation using a single sigma, but we should start the index from $n = 1$. Therefore we have to write $n = 0$ terms separately.

$$ 2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + (2n+2)a_n] x^n = 0 $$

$$ a_2 = -a_0, \quad a_{n+2} = \frac{-2(n+1)}{(n+2)(n+1)} a_n = \frac{-2}{(n+2)} a_n $$

This is called the recursion relation. Using it, we can find all the constants in terms of $a_0$ and $a_1$.

$$ a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0, \quad a_6 = -\frac{2}{6}a_4 = -\frac{1}{6}a_0 $$

$$ a_3 = -\frac{2}{3}a_1, \quad a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1 $$

We can find as many coefficients as we want in this way. Collecting them together, the solution is:

$$ y = a_0 \left( 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots \right) + a_1 \left( x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots \right) $$
In most applications, we want a solution close to 0, therefore we can neglect the higher order terms of the series.

**Remark:** Sometimes we can express the solution in closed form (in terms of elementary functions rather than an infinite summation) as in the next example:

**Example 6.2** Solve $$(x - 1)y'' + 2y' = 0$$ around $x_0 = 0$.

Once again, first we should classify the given point. The function $\frac{2}{x - 1}$ is analytic at $x = 0$, therefore $x = 0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2}$$

Inserting these in the equation, we obtain

$$(x - 1) \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

To equate the powers of $x$, let us replace $n$ by $n + 1$ in the second summation.

$$\sum_{n=2}^{\infty} n(n - 1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} = 0$$

Now we can express the equation using a single sigma.

$$(-2a_2 + 2a_1) + \sum_{n=2}^{\infty} [(n(n - 1) + 2n)a_n - n(n + 1)a_{n+1}] x^{n-1} = 0$$

$$a_2 = a_1, \quad a_{n+1} = \frac{n^2 - n + 2n}{n(n + 1)} a_n \text{ for } n \geq 2$$

So the recursion relation is:

$$a_{n+1} = a_n$$

All the coefficients are equal to $a_1$, except $a_0$. We have no information about it, so it must be arbitrary. Therefore, the solution is:

$$y = a_0 + a_1 \left( x + x^2 + x^3 + \cdots \right)$$

$$y = a_0 + a_1 \frac{x}{1 - x}$$
Exercises

Find the general solution of the following differential equations in the form of series. Find solutions around the origin (use $x_0 = 0$). Write the solution in closed form if possible.

1) $(1 - x^2)y'' - 2xy' = 0$
2) $y'' + x^4y' + 4x^3y = 0$
3) $(2 + x^3)y'' + 6x^2y' + 6xy = 0$
4) $(1 + x^2)y'' - xy' - 3y = 0$
5) $(1 + 2x^2)y'' + xy' + 2y = 0$
6) $y'' - xy' + ky = 0$
7) $(1 + x^2)y'' - 4xy' + 6y = 0$
8) $(1 - 2x^2)y'' + (2x + 4x^3)y' - (2 + 4x^2)y = 0$
9) $(1 + 8x^2)y'' - 16y = 0$
10) $y'' + x^2y = 0$

The following equations give certain special functions that are very important in applications. Solve them for $n = 1, 2, 3$ around origin. Find polynomial solutions only.

11) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ (Legendre’s Equation)
12) $y'' - 2xy' + 2ny = 0$ (Hermite’s Equation)
13) $xy'' + (1 - x)y' + ny = 0$ (Laguerre’s Equation)
14) $(1 - x^2)y'' - xy' + n^2y = 0$ (Chebyshev’s Equation)

Solve the following initial value problems. Find the solution around the point where initial conditions are given.

15) $xy'' + (x + 1)y' - 2y = 0$, $x_0 = -1$, $y(-1) = 1$, $y'(-1) = 0$
16) $y'' + 2xy' - 4y = 0$, $x_0 = 0$, $y(0) = 1$, $y'(0) = 0$
17) $4y'' + 3xy' - 6y = 0$, $x_0 = 0$, $y(0) = 4$, $y'(0) = 0$
18) $(x^2 - 4x + 7)y'' + y = 0$, $x_0 = 2$, $y(2) = 4$, $y'(2) = 10$

19) Find the recursion relation for $(p + x^2)y'' + (1 - q - r)xy' + qry = 0$ around $x = 0$. (Here $p, q, r$ are real numbers, $p \neq 0$)
20) Solve $(1 + ox^2)y'' + bxy' + cy = 0$ around $x_0 = 0$
Answers

1) \( y = a_0 + a_1 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) \) OR \( y = a_0 + a_1 \left( \frac{1}{2} \ln \frac{1+x}{1-x} \right) \)

2) \( y = a_0 \left( 1 - \frac{x^5}{5} + \frac{x^{10}}{5 \cdot 10} - \frac{x^{15}}{5 \cdot 10 \cdot 15} + \cdots \right) \\
    + a_1 \left( x - \frac{x^6}{6} + \frac{x^{11}}{6 \cdot 11} - \frac{x^{16}}{6 \cdot 11 \cdot 16} + \cdots \right) \)

3) \( y = a_0 \left( 1 - \frac{x^3}{3} + \frac{x^6}{4} - \frac{x^9}{8} + \cdots \right) + a_1 \left( x - \frac{x^4}{2} + \frac{x^7}{4} - \frac{x^{10}}{8} + \cdots \right) \) OR \( y = \frac{a_0}{1 + \frac{x^3}{2}} + \frac{a_1 x^3}{1 + \frac{x^3}{2}} \)

4) \( y = a_0 \left( 1 + \frac{3}{2} x^2 + \frac{3}{8} x^4 - \frac{1}{16} x^6 + \cdots \right) + a_1 \left( x + \frac{2}{3} x^3 \right) \)

5) \( y = a_0 \left( 1 - x^2 + \frac{2}{3} x^4 - \frac{2}{3} x^6 + \cdots \right) + a_1 \left( x - \frac{1}{2} x^3 + \frac{17}{40} x^5 + \cdots \right) \)

6) \( y = a_0 \left[ 1 - \frac{k}{2} x^2 + \frac{k(k-2)}{4!} x^4 - \frac{k(k-2)(k-4)}{6!} x^6 + \cdots \right] \\
    + a_1 \left[ x - \frac{k-1}{3!} x^3 + \frac{(k-1)(k-3)}{5!} x^5 - \frac{(k-1)(k-3)(k-5)}{7!} x^7 + \cdots \right] \)

7) \( y = a_0 (1 - 3x^2) + a_1 \left( x - \frac{x^3}{3} \right) \)

8) \( y = a_0 \left( 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots \right) + a_1 x \)

9) \( y = a_0 (1 + 8x^2) + a_1 \left( x + \frac{8}{3} x^3 - \frac{64}{15} x^5 + \cdots \right) \)

10) \( y = a_0 \left( 1 - \frac{x^4}{12} + \frac{x^8}{672} + \cdots \right) + a_1 \left( x - \frac{x^5}{20} + \frac{x^9}{1440} + \cdots \right) \)

11) \( n = 1 \Rightarrow y = a_1 x \\
    n = 2 \Rightarrow y = a_0 (1 - 3x^2) \\
    n = 3 \Rightarrow y = a_1 (x - \frac{5}{3} x^3) \)
12) \( n = 1 \Rightarrow y = a_1 x \)
\( n = 2 \Rightarrow y = a_0 (1 - 2x^2) \)
\( n = 3 \Rightarrow y = a_1 (x - \frac{2}{3}x^3) \)

13) \( n = 1 \Rightarrow y = a_0 (1 - x) \)
\( n = 2 \Rightarrow y = a_0 (1 - 2x + \frac{1}{2}x^2) \)
\( n = 3 \Rightarrow y = a_0 (1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3) \)

14) \( n = 1 \Rightarrow y = a_1 x \)
\( n = 2 \Rightarrow y = a_0 (1 - 2x^2) \)
\( n = 3 \Rightarrow y = a_1 (x - \frac{4}{3}x^3) \)

15) \( y = 1 - (x + 1)^2 - \frac{1}{3}(x + 1)^3 - \frac{1}{6}(x + 1)^4 - \cdots \)

16) \( y = 1 + 2x^2 \)

17) \( y = 4 + 3x^2 \)

18) \( y = 4 \left[ 1 - \frac{1}{6}(x - 2)^2 + \frac{1}{72}(x - 2)^4 + \cdots \right] \)
\( + 10 \left[ (x - 2) - \frac{1}{18}(x - 2)^3 + \frac{7}{1080}(x - 2)^5 + \cdots \right] \)

19) \( a_{n+2} = -\frac{(n-q)(n-r)}{p(n+2)(n+1)} a_n \)

20) \( y = a_0 \left[ 1 - c \frac{x^2}{2} + c(2a + 2b + c) \frac{x^4}{4!} \right. \)
\( - c(2a + 2b + c)(12a + 4b + c) \frac{x^6}{6!} + \cdots \]
\( + a_1 \left[ x - (b + c) \frac{x^3}{3!} + (b + c)(6a + 3b + c) \frac{x^5}{5!} \right. \)
\( - (b + c)(6a + 3b + c)(20a + 5b + c) \frac{x^7}{7!} + \cdots \]
Chapter 7

Frobenius’ Method

In this chapter, we will extend the methods of the previous chapter to regular singular points. The calculations will be considerably longer, but the basic ideas are the same. The classification of the given point is necessary to make a choice of methods.

7.1 An Extension of Power Series Method

Suppose $x_0$ is a regular singular point. For simplicity, assume $x_0 = 0$. Then the differential equation can be written as $y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$ where $p(x)$ and $q(x)$ are analytic. We can try a solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \quad (7.1)$$

The equation corresponding to the lowest power $x^{r-2}$, in other words $r(r - 1) + p_0 r + q_0 = 0$ is called the indicial equation, where $p_0 = p(0)$, and $q_0 = q(0)$. Now we can find $r$, insert it in the series formula, and proceed as we did in the previous chapter.

We can classify the solutions according to the roots of the indicial equation.

**Case 1 - Distinct roots not differing by an integer:** A basis of solutions is

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad (7.2)$$
Case 2 - Equal roots: A basis of solutions is
\[ y_1 = x^r \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = y_1 \ln x + x^r \sum_{n=1}^{\infty} b_n x^n \] (7.3)

Case 3 - Roots differing by an integer: A basis of solutions is
\[ y_1 = x^{r_1} \sum_{n=1}^{\infty} a_n x^n, \quad y_2 = ky_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \] (7.4)

where \( r_1 - r_2 = N > 0 \) (\( r_1 \) is the greater root) and \( k \) may or may not be zero.

In all three cases, there is at least one relatively simple solution of the form \( y = x^r \sum_{n=0}^{\infty} a_n x^n \). The equation is second order, so there must be a second linearly independent solution. In Cases 2 and 3, it may be difficult to find the second solution. You may use the method of reduction of order. This is convenient especially if \( y_1 \) is simple enough. Alternatively, you may use the above formulas directly, and determine \( b_n \) one by one using the \( a_n \) and the equation.

### 7.2 Examples

**Example 7.1** Solve \( 4xy'' + 2y' + y = 0 \) around \( x_0 = 0 \).

First we should classify the given point. The function \( \frac{2}{x^4} \) is not analytic at \( x = 0 \) therefore \( x = 0 \) is a singular point. We should make a further test to determine whether it is regular or not.

The functions \( \frac{2}{x^4} \) and \( \frac{x^2}{2} \) are analytic therefore \( x = 0 \) is a R.S.P., we can use the method of Frobenius.

\[
y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}\]

Note that the summation for the derivatives still starts from 0, because \( r \) does not have to be an integer. This is an important difference between methods of power series and Frobenius.

Inserting these in the equation, we obtain
\[
4x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\]
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\[
\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\]

We want to equate the powers of \( x \), so \( n \to n+1 \) in the first two terms.

\[
\sum_{n=-1}^{\infty} 4(n+r+1)(n+r)a_{n+1} x^{n+r+1} + \sum_{n=-1}^{\infty} 2(n+r+1)a_{n+1} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\]

Now we can express the equation using a single sigma, but the index of the common sigma must start from \( n = 0 \). Therefore we have to write \( n = -1 \) terms separately.

\[
[4r(r-1)+2r]a_0 x^{r-1} + \sum_{n=0}^{\infty} \{[4(n+r+1)(n+r) + 2(n+r+1)]a_{n+1} + a_n\} x^{n+r} = 0
\]

We know that \( a_0 \neq 0 \), therefore \( 4r^2 - 2r = 0 \). This is the indicial equation.

Its solutions are \( r = 0, r = \frac{1}{2} \). Therefore this is Case 1.

If \( r = 0 \), the recursion relation is

\[
a_{n+1} = \frac{-1}{4(n+1)(n+\frac{1}{2})} a_n
\]

\[
a_1 = -\frac{a_0}{2}, \quad a_2 = -\frac{a_1}{4.2^{\frac{1}{2}}} = \frac{a_0}{4!}, \quad a_3 = -\frac{a_2}{4.3.\frac{1}{2}} = -\frac{a_0}{6!}, \ldots
\]

For simplicity, we may choose \( a_0 = 1 \). Then

\[
a_n = \frac{(-1)^n}{2n!}
\]

Therefore the first solution is:

\[
y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n!} = \cos \sqrt{x}
\]

If \( r = \frac{1}{2} \), the recursion relation is

\[
a_{n+1} = \frac{-1}{4(n+\frac{3}{2})(n+1)} a_n = \frac{-a_n}{(2n+3)(2n+2)}
\]

\[
a_1 = -\frac{a_0}{3.2}, \quad a_2 = -\frac{a_1}{5.4} = \frac{a_0}{5!}, \quad a_3 = -\frac{a_2}{7.6} = -\frac{a_0}{7!}, \ldots
\]
For simplicity, we may choose \( a_0 = 1 \). Then
\[
a_n = \frac{(-1)^n}{(2n + 1)!}
\]
Therefore the second solution is:
\[
y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n + 1)!} = \sin \sqrt{x}
\]
The general solution is \( y = c_1 y_1 + c_2 y_2 \)

**Example 7.2** Solve \( x^2 y'' + (x^2 - x)y' + (1 + x)y = 0 \) around \( x_0 = 0 \).

First we should classify the given point. The function \( \frac{x^2 - x}{x} \) is not analytic at \( x = 0 \) therefore \( x = 0 \) is a singular point. The functions \( x - 1 \) and \( 1 + x \) are analytic at \( x = 0 \) therefore \( x = 0 \) is a R.S.P., we can use the method of Frobenius. Evaluating the derivatives of \( y \) and inserting them in the equation, we obtain
\[
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n + r)a_n x^{n+r+1} \\
- \sum_{n=0}^{\infty} (n + r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0
\]
Let’s replace \( n \) by \( n - 1 \) in the second and fifth terms.
\[
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r} + \sum_{n=1}^{\infty} (n + r - 1)a_{n-1} x^{n+r} \\
- \sum_{n=0}^{\infty} (n + r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0
\]
\[
[r^2 - 2r + 1]a_0 x^r + \\
\sum_{n=1}^{\infty} \left\{ [(n + r)(n + r - 1) - (n + r) + 1]a_n + [(n + r - 1) + 1]a_{n-1} \right\} x^{n+r} = 0
\]
The indicial equation is \( r^2 - 2r + 1 = 0 \) \( \Rightarrow r = 1 \) (double root). Therefore this is Case 2. The recursion relation is
\[
a_n = -\frac{n + 1}{n^2} a_{n-1}
\]
For simplicity, let $a_0 = 1$. Then

\[ a_1 = -2, \quad a_2 = -\frac{3}{4}a_1 = -\frac{3}{2}, \quad a_3 = -\frac{4}{9}a_2 = -\frac{2}{3} \]

Therefore the first solution is:

\[ y_1 = x \left( 1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \cdots \right) \]

To find the second solution, we will use reduction of order. Let $y_2 = uy_1$.

Inserting $y_2$ in the equation, we obtain

\[ x^2 y_1 u'' + (2x^2 y_1' - xy_1 + x^2 y_1) u' = 0 \]

Let $w = u'$ then

\[ w' + \left( \frac{2y_1'}{y_1} - \frac{1}{x} + 1 \right) w = 0 \]

\[ \frac{dw}{w} = \left( -2 \frac{y_1'}{y_1} + \frac{1}{x} - 1 \right) dx \]

\[ \ln w = -2 \ln y_1 + \ln x - x \quad \Rightarrow \quad w = \frac{xe^{-x}}{y_1^2} \]

To evaluate the integral $u = \int w \, dx$ we need to find $\frac{1}{y_1^2}$. This is also a series.

\[ \frac{1}{y_1^2} = \frac{1}{x^2} \left( 1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \cdots \right)^{-2} = \frac{1}{x^2} \left( 1 + 4x + 9x^2 + \frac{46}{3}x^3 + \cdots \right) \]

\[ w = \frac{xe^{-x}}{y_1^2} = x \left( 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots \right) \frac{1}{x^2} \left( 1 + 4x + 9x^2 + \frac{46}{3}x^3 + \cdots \right) \]

\[ w = \frac{1}{x} \left( 1 + 3x + \frac{11}{2}x^2 + \frac{13}{6}x^3 + \cdots \right) \]

\[ u = \int w \, dx = \ln x + 3x + \frac{11}{4}x^2 + \frac{13}{18}x^3 + \cdots \]

\[ y_2 = uy_1 = y_1 \ln x + x \left( 3x - \frac{13}{4}x^2 + \frac{3}{2}x^3 + \cdots \right) \]
Exercises

Find two linearly independent solutions of the following differential equations in the form of series. Find solutions around the origin (use $x_0 = 0$). Write the solution in closed form if possible.

1) $2x^2 y'' - x y' + (1 + x) y = 0$
2) $2xy'' + (1 + x)y' - 2y = 0$
3) $(x^2 + 2x)y'' + (3x + 1)y' + y = 0$
4) $xy'' - y' - 4x^3 y = 0$
5) $xy'' + y' - xy = 0$
6) $3x^2 y'' + (-10x - 3x^2)y' + (14 + 4x)y = 0$
7) $x^2 y'' + (x^2 - x)y' + y = 0$
8) $(2x^2 + 2x)y'' - y' - 4y = 0$
9) $2x^2 y'' + (2x^2 - x)y' + y = 0$
10) $4x^2 y'' + (2x^2 - 10x)y' + (12 - x)y = 0$
11) $(x^2 + 2x)y'' + (4x + 1)y' + 2y = 0$

Use Frobenius’ method to solve the following differential equations around origin. Find the roots of the indicial equation, find the recursion relation, and two linearly independent solutions.

12) $(x^2 + cx)y'' + [(2 + b)x + c(1 - d)]y' + by = 0$
   \quad (b \neq 0, c \neq 0, d \text{ is not an integer}).

13) $x^2 y'' + [(1 - b - d)x + cx^2]y' + [bd + (1 - b)cx]y = 0$
   \quad (c \neq 0, b - d \text{ is not an integer}).

14) $x^2 y'' + [(1 - 2d)x + cx^2]y' + (d^2 + (1 - d)cx)y = 0$
   \quad (c \neq 0)$

15) $xy'' + [1 - d + cx^2]y' + 2cxy = 0$
   \quad (c \neq 0, d \text{ is not an integer}).
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Answers

1) \[ y = c_1 x \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)} \right) + c_2 x^{\frac{1}{2}} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)} \right) \]

2) \[ y = c_1 \left( 1 + 2x + \frac{x^3}{3} x^2 + 1 \right) + c_2 x^{\frac{1}{2}} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 3x^n}{2^n n! (2n-3)(2n-1)(2n+1)} \right) \]

3) \[ y_1 = 1 - x + \frac{2}{3} x^2 - \frac{6}{15} x^3 + \cdots, \quad y_2 = x^{\frac{1}{2}} \left( 1 - \frac{3}{4} x + \frac{15}{32} x^2 - \frac{35}{128} x^3 + \cdots \right) \]

4) \[ y = a_0 \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!} + a_2 \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}, \quad \text{OR} \quad y = c_1 e^{x^2} + c_2 e^{-x^2} \]

5) \[ y_1 = 1 + \frac{x^2}{2^2} + \frac{x^4}{(2 \cdot 4)^2} + \frac{x^6}{(2 \cdot 4 \cdot 6)^2} + \cdots, \quad y_2 = \frac{x}{2} \frac{1}{\ln x} - \frac{x^2}{4} - \frac{3x^4}{8 \cdot 16} - \frac{11x^6}{64 \cdot 6 \cdot 36} - \cdots \]

6) \[ y_1 = x^{7/3} \left( 1 + \frac{3}{4} x + \frac{9}{28} x^2 + \frac{27}{280} x^3 + \cdots \right) \]

\[ y_2 = x^2 \left( 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) = x^2 e^x \]

7) \[ y_1 = x \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right) = xe^{-x} \]

\[ y_2 = xe^{-x} \ln x + xe^{-x} \left( x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \cdots \right) \]

8) \[ y_1 = 1 - 4x - 8x^2, \quad y_2 = x^{3/2} \left( 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 + \cdots \right) \]

9) \[ y_1 = x^{1/2} e^{-x}, \quad y_2 = x \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \right] \]

10) \[ y_1 = x^{2} e^{-x/2}, \quad y_2 = x^{3/2} \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right] \]
CHAPTER 7. FROBENIUS’ METHOD

11) \[ y_1 = 1 - 2x + 2x^2 - \frac{8}{5}x^3 + \cdots \]
\[ y_2 = x^{1/2} \left( 1 - \frac{5}{4}x + \frac{35}{32}x^2 - \frac{105}{128}x^3 + \cdots \right) \]

12) \[ r = 0 \Rightarrow a_{n+1} = -\frac{n + b}{c(n + 1 - d)} a_n \]
\[ y_1 = 1 - \frac{b}{c(1 - d)} x + \frac{b(b + 1)}{c^2(1 - d)(2 - d)} x^2 - \cdots \]
\[ r = d \Rightarrow a_{n+1} = -\frac{n + b + d}{c(n + 1)} a_n \]
\[ y_2 = x^d \left[ 1 - \frac{d + b}{c} x + \frac{(d + b)(d + b + 1)}{2!c^2} x^2 - \cdots \right] \]

13) \[ r = b \Rightarrow a_n = -\frac{c}{n + b - d} a_{n-1} \]
\[ y_1 = x^b \left[ 1 - \frac{c}{1 + b - d} x + \frac{c^2}{(1 + b - d)(2 + b - d)} x^2 - \cdots \right] \]
\[ r = d \Rightarrow a_n = -\frac{c}{n} a_{n-1} \]
\[ y_2 = x^d \left( 1 - cx + \frac{c^2}{2!} x^2 - \frac{c^3}{3!} x^3 + \cdots \right) = x^d e^{-cx} \]

14) \[ r = d \text{ (double root)} \ a_n = -\frac{c}{n} a_{n-1} \]
\[ y_1 = x^d \left( 1 - cx + \frac{c^2}{2!} x^2 - \frac{c^3}{3!} x^3 + \cdots \right) = x^d e^{-cx} \]
\[ y_2 = x^d e^{-cx} \frac{e^x}{x} \frac{d}{dx} \]
\[ y_2 = x^d e^{-cx} \ln x + x^d e^{-cx} \left( cx + \frac{c^2}{2 \cdot 2!} x^2 + \frac{c^3}{3 \cdot 3!} x^3 + \cdots \right) \]

15) \[ r = 0 \Rightarrow a_{n+2} = -\frac{c}{(n + 2 - d)} a_n \]
\[ y_1 = 1 - \frac{c}{2 - d} x^2 + \frac{c^2}{(2 - d)(4 - d)} x^4 - \frac{c^3}{(2 - d)(4 - d)(6 - d)} x^6 + \cdots \]
\[ r = d \Rightarrow a_{n+2} = -\frac{c}{n + 2} a_n \]
\[ y_2 = x^d \left( 1 - \frac{c}{2} x^2 + \frac{c^2}{2 \cdot 4} x^4 - \frac{c^3}{2 \cdot 4 \cdot 6} x^6 + \cdots \right) \]
Chapter 8

Laplace Transform I

Laplace transform provides an alternative method for many equations. We first transform the differential equation to an algebraic equation, then solve it, and then make an inverse transform. Laplace transform has a lot of interesting properties that make these operations easy. In this chapter, we will see the definition and the basic properties. We will also compare this method to the method of undetermined coefficients, and see in what ways Laplace transform is more convenient.

8.1 Definition, Existence and Inverse of Laplace Transform

The Laplace transform of a function \( f(t) \) is defined as:

\[
F(s) = \mathcal{L} \{ f(t) \} = \int_{0}^{\infty} e^{-st} f(t) dt
\]

(8.1)

then, the inverse transform will be

\[
f(t) = \mathcal{L}^{-1} \{ F(s) \}
\]

(8.2)

Note that we use lowercase letters for functions and capital letters for their transforms.
Example 8.1 Evaluate the Laplace transform of the following functions:

(a) \( f(t) = 1 \)
\[
\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \left. \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}, \quad s > 0
\]

(b) \( f(t) = e^{at} \)
\[
\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st} \, dt = \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty = \frac{1}{s-a}, \quad s > a
\]

(c) \( f(t) = \begin{cases} 
0 & \text{if } 0 < t < 1 \\
1 & \text{if } 1 \leq t
\end{cases} \)
\[
\mathcal{L}\{f\} = \int_1^\infty e^{-st} \, dt = \left. \frac{e^{-st}}{-s} \right|_1^\infty = \frac{e^{-s}}{s}, \quad s > 0
\]

(d) \( f(t) = t \)
\[
\mathcal{L}\{t\} = \int_0^\infty te^{-st} \, dt
\]
Using integration by parts, we obtain
\[
\mathcal{L}\{t\} = \left. -t \frac{e^{-st}}{-s} \right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} \, dt
\]
\[
\mathcal{L}\{t\} = \left. -\frac{e^{-st}}{s^2} \right|_0^\infty = \frac{1}{s^2}, \quad s > 0
\]

The integral that defines the Laplace transform is an improper integral, it may or may not converge. In the above examples, the transform is defined for a certain range of \( s \).

In practice, we can use Laplace transform on most of the functions we encounter in differential equations. The following definitions and the theorem answer the question Which functions have a Laplace transform?

**Piecewise Continuous Functions:** A function \( f(t) \) is piecewise continuous on \([a, b]\) if the interval can be subdivided into subintervals \([t_i, t_j]\), \( a = t_0 < t_1 < t_2 \cdots < t_n = b \) such that \( f(t) \) is continuous on each interval and has finite one-sided limits at the endpoints (from the interior).

An example can be seen on Figure 8.1.
Exponential Order: \( f(t) \) is of exponential order as \( t \to \infty \) if there exist real constants \( M, c, T \) such that \( |f(t)| \leq Me^{ct} \) for all \( t \geq T \). In other words, a function is of exponential order if it does not grow faster than \( e^{ct} \).

**Theorem 8.1:** If \( f(t) \) is of exponential order and piecewise continuous on \([0, k]\) for all \( k > 0 \), then its Laplace transform exists for all \( s > c \).

For example, all the polynomials have a Laplace transform. The function \( e^{t^2} \) does NOT have a Laplace transform.

### 8.2 Basic Properties of Laplace Transforms

It is difficult to evaluate the Laplace transform of each function by performing an integration. Instead of this, we use various properties of Laplace transform.

Let \( \mathcal{L}\{f(t)\} = F(s) \), then, some basic properties are: (assuming these transforms exists)

- **Linearity**
  \[
  \mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}
  \]

- **Shifting**
  \[
  \mathcal{L}\{e^{at}f(t)\} = F(s - a)
  \]
  \[
  \mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)
  \]
• Transform of Derivatives

\[ \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0) \]
\[ \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0) \]
\[ \mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \]

• Transform of Integrals

\[ \mathcal{L}\{\int_0^t f(x)\,dx\} = \frac{F(s)}{s} \]

**Example 8.2** Find the Laplace transform of \( \sin at \) and \( \cos at \). Hint: Use Euler’s formula \( e^{ix} = \cos x + i\sin x \) and linearity.

\[ \sin at = \frac{e^{iat} - e^{-iat}}{2i} \quad \Rightarrow \quad \mathcal{L}\{\sin at\} = \frac{\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}}{2i} \]

\[ \mathcal{L}\{\sin at\} = \frac{1}{2i} \left( \frac{1}{s - ia} - \frac{1}{s + ia} \right) = \frac{a}{s^2 + a^2} \]

Similarly, we can show that the transform of \( f(t) = \cos at \) is

\[ F(s) = \frac{s}{s^2 + a^2} \]

**Example 8.3** Find the inverse Laplace transform of \( F(s) = \frac{1}{(s + 5)^2} \). Hint: Use shifting.

We know that \( \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} = t \). Therefore

\[ \mathcal{L}^{-1}\left\{ \frac{1}{(s + 5)^2} \right\} = te^{-5t} \]

**Example 8.4** Find the Laplace transform of \( f(t) = t^2 \). Hint: Use Derivatives.

Using \( \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0) \), we obtain

\[ \mathcal{L}\{2t\} = s\mathcal{L}\{t^2\} - 0 \quad \Rightarrow \quad \mathcal{L}\{t^2\} = \frac{\mathcal{L}\{2t\}}{s} = \frac{2}{s^3} \]

**Example 8.5** Find the Laplace transform of \( f(t) = t^4 \). Hint: Use Integrals.

Using the integral rule, we see that

\[ \mathcal{L}\left\{ \frac{t^3}{3} \right\} = \mathcal{L}\{t^2\} = \frac{2}{s^4} \]

\[ \mathcal{L}\{t^3\} = \frac{6}{s^4} \]
8.3 Initial Value Problems

Consider the constant-coefficient equation

\[ y'' + ay' + by = r(t) \]  \hspace{1cm} (8.3)

with initial values

\[ y(0) = p, \ y'(0) = q \]  \hspace{1cm} (8.4)

Here \( y \) is a function of \( t \) \((y = y(t))\). We can solve it by the method of undetermined coefficients. The method of Laplace transform will be an alternative that is more efficient in certain cases. It also works for discontinuous \( r(t) \).

Let us evaluate the Laplace transform of both sides.

\[ \mathcal{L}\{ y'' \} + a \mathcal{L}\{ y' \} + b \mathcal{L}\{ y \} = \mathcal{L}\{ r(t) \} \]  \hspace{1cm} (8.5)

Using \( \mathcal{L}\{ y \} = Y(s) \) and \( \mathcal{L}\{ r(t) \} = R(s) \)

\[ s^2Y - sp - q + a(sY - p) + bY = R \]  \hspace{1cm} (8.6)

\[ (s^2 + as + b)Y = R + (s + a)p + q \]  \hspace{1cm} (8.7)

\[ Y = \frac{R + (s + a)p + q}{s^2 + as + b} \]  \hspace{1cm} (8.8)

\[ y = \mathcal{L}^{-1}\left\{ \frac{R + sp + ap + q}{s^2 + as + b} \right\} \]  \hspace{1cm} (8.9)

Note that this method can be generalized to higher order equations. The advantages compared to the method of undetermined coefficients are:

- The initial conditions are built in the solution, we don’t need to determine constants after obtaining the general solution.

- There is no distinction between homogeneous and nonhomogeneous equations, or single and multiple roots. The same method works in all cases the same way.

- The function on the right hand side \( r(t) \) belongs to a wider class. For example, it can be discontinuous.
The only disadvantage is that, sometimes finding the inverse Laplace transform is too difficult.

We have to find roots of the polynomial \( s^2 + as + b \), which is the same as the characteristic polynomial we would encounter if we were using method of undetermined coefficients.

**Example 8.6** Solve the initial value problem

\[ y'' + 4y = 0, \; y(0) = 5, \; y'(0) = 3. \]

Let’s start by finding the transform of the equation.

\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 0
\]

\[
s^2Y - 5s - 3 + 4Y = 0 \quad \Rightarrow \quad (s^2 + 4)Y = 5s + 3
\]

\[
Y = \frac{5s + 3}{s^2 + 4}
\]

Now, we have to find the inverse transform of \( Y \) to obtain \( y(t) \).

\[
Y = \frac{5s}{s^2 + 4} + \frac{3}{2} \frac{2}{s^2 + 4}
\]

\[
y(t) = \mathcal{L}^{-1}\{Y\} = 5\cos 2t + \frac{3}{2} \sin 2t
\]

Note that we did not first find the general solution containing arbitrary constants. We directly found the result.

**Example 8.7** Solve the initial value problem

\[ y'' - 4y' + 3y = 1, \; y(0) = 0, \; y'(0) = -\frac{1}{3} \]

Transform both sides:

\[
\mathcal{L}\{y'' - 4y' + 3y\} = \mathcal{L}\{1\}
\]

Use the derivative rule

\[
s^2Y - s.0 + \frac{1}{3} - 4(sY - 0) + 3Y = \frac{1}{s}
\]
8.3. INITIAL VALUE PROBLEMS

Isolate $Y$

$$(s^2 - 4s + 3)Y = \frac{1}{s} - \frac{1}{3} = \frac{3 - s}{3s}$$

$$(s - 1)(s - 3)Y = -\frac{s - 3}{3s}$$

$$Y = -\frac{1}{3s(s - 1)} = \frac{1}{3} \left( \frac{1}{s} - \frac{1}{s - 1} \right)$$

Find the inverse transform

$$y(t) = \mathcal{L}^{-1} \{ Y \} = \frac{1}{3} - \frac{1}{3} e^t$$

As you can see, there’s no difference between homogeneous and nonhomogeneous equations. Laplace transform works for both types in the same way.

**Example 8.8** Solve the initial value problem

$$y'' + 4y' + 4y = 42te^{-2t}, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L} \{ y'' \} + 4\mathcal{L} \{ y' \} + 4\mathcal{L} \{ y \} = 42\mathcal{L} \{ te^{-2t} \}$$

$$s^2Y + 4sY + 4Y = \frac{42}{(s + 2)^2}$$

$$Y = \frac{42}{(s + 2)^4}$$

$$y(t) = \mathcal{L}^{-1} \{ Y(s) \} = \frac{42}{3!} t^3 e^{-2t}$$

$$y(t) = 7t^3 e^{-2t}$$

If you try the method of undetermined coefficients on this problem, you will appreciate the efficiency of Laplace transforms better.
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<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$e^{at} \sin bt$</td>
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<td>$\sin at - at \cos at$</td>
<td>$\frac{2a^3}{(s^2 + a^2)^2}$</td>
<td>$\sin at + at \cos at$</td>
<td>$\frac{2as^2}{(s^2 + a^2)^2}$</td>
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Table 8.1: A Table of Laplace Transforms
Exercises

Find the Laplace transform of the following functions:

1) \( f(t) = \cos^2 \frac{t}{2} \)
2) \( f(t) = e^t \sin 3t \)
3) \( f(t) = 2e^{-t} \cos^2 t \)
4) \( f(t) = (t + 1)^2 e^t \)
5) \( f(t) = t^3 e^{3t} \)
6) \( f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & a < t \end{cases} \)
7) \( f(t) = \begin{cases} t & 0 < t < a \\ 0 & a < t \end{cases} \)
8) \( f(t) = \begin{cases} t & 0 < t < a \\ 1 & a < t < b \\ 0 & b < t \end{cases} \)

Find the inverse Laplace transform of the following functions:

9) \( F(s) = \frac{s^2 - 4}{s^2 - 4} \)
10) \( F(s) = \frac{3}{(s - 2)^2} \)
11) \( F(s) = \frac{6}{s(s + 4)} \)
12) \( F(s) = \frac{1}{s(s^2 + 9)} \)
13) \( F(s) = \frac{1}{s^2(s + 1)} \)
14) \( F(s) = \frac{5s + 1}{s^2 + 4} \)
15) \( F(s) = \frac{1}{s + 8} \)
16) \( F(s) = \frac{1}{(s - a)^n} \)

Solve the following initial value problems using Laplace transform:

17) \( y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3 \)
18) \( y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1 \)
19) \( y'' + 2y = 4t^2 + 12, \quad y(0) = 4, \quad y'(0) = 0 \)
20) \( y'' + 6y' + 9y = e^{-3t}, \quad y(0) = 0, \quad y'(0) = 0 \)
Answers

1) \( F(s) = \frac{1}{2s} + \frac{s}{2s^2 + 2} \)

2) \( F(s) = \frac{3}{(s - 1)^2 + 9} \)

3) \( F(s) = \frac{1}{s + 1} + \frac{s + 1}{s^2 + 2s + 5} \)

4) \( F(s) = \frac{2}{(s - 1)^3} + \frac{2}{(s - 1)^2} + \frac{1}{s - 1} \)

5) \( F(s) = \frac{6}{(s - 3)^4} \)

6) \( F(s) = \frac{1}{s} - e^{-as} \)

7) \( F(s) = \frac{1}{s^2} - \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} \)

8) \( F(s) = \frac{1}{s^2} - \frac{e^{-as} - ae^{-as} - e^{-bs}}{s} \)

9) \( f(t) = \cosh 2t - 2\sinh 2t \)

10) \( f(t) = 3te^{2t} \)

11) \( f(t) = (3 - 3e^{-4t})/2 \)

12) \( f(t) = (1 - \cos 3t)/9 \)

13) \( f(t) = e^{-t} + t - 1 \)

14) \( f(t) = 5\cos 2t + \frac{1}{2}\sin 2t \)

15) \( f(t) = e^{-8t} \)

16) \( f(t) = \frac{t^{n-1}e^{at}}{(n-1)!} \)

17) \( g(t) = 4e^t - 7te^t \)

18) \( g(t) = e^t \sin t \)

19) \( g(t) = 4 + 2t^2 \)

20) \( g(t) = \frac{1}{2}e^{-3t}t^2 \)
Chapter 9

Laplace Transform II

In this chapter, we will study more advanced properties of Laplace transform. At the end, we will be able to find transform and inverse transform of a wider range of functions. This will enable us to solve almost any linear constant coefficient equation, including discontinuous inputs.

9.1 Convolution

The convolution of two functions $f$ and $g$ is defined as

$$h(t) = (f * g)(t) = \int_0^t f(x)g(t-x)\,dx$$  \hfill (9.1)

The convolution operation is commutative, in other words $f * g = g * f$

**Theorem 9.1:** The transform of convolution of two functions is equal to the product of their transforms, i.e.

$$L\{f * g\} = F(s) \cdot G(s)$$  \hfill (9.2)

$$L^{-1}\{F(s) \cdot G(s)\} = f * g$$  \hfill (9.3)

where $L\{f\} = F(s)$ and $L\{g\} = G(s)$.

**Proof:** Using the definitions of convolution and Laplace transform,

$$L\{f * g\} = L\left\{ \int_0^t f(x)g(t-x)\,dx \right\}$$

$$= \int_0^\infty \int_0^t f(x)g(t-x)e^{-st}\,dx\,dt$$
Reversing the order of integration, we obtain:

\[ = \int_0^\infty \int_0^\infty f(x) g(t-x) e^{-st} \, dx \, dt \]

Making the substitution \( u = t - x \), we obtain:

\[ \mathcal{L}\{f * g\} = \int_0^\infty \int_0^\infty f(x) g(u) e^{-su-sx} \, dx \, du = \int_0^\infty f(x)e^{-sx} \, dx \int_0^\infty g(u)e^{-su} \, du = F(s) G(s) \]

**Example 9.1** Find the inverse Laplace transform of \( F(s) = \frac{1}{s^3 + 4s^2} \).

\[ L^{-1}\left\{ \frac{1}{s^2} \right\} = t, \quad L^{-1}\left\{ \frac{1}{s+4} \right\} = e^{-4t} \Rightarrow L^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{s+4} \right\} = t * e^{-4t} \]

\[ f(t) = t * e^{-4t} = \int_0^t x e^{-4(t-x)} \, dx = e^{-4t} \left( \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} \right) \bigg|_0^t = \frac{t}{4} - \frac{1}{16} + \frac{e^{-4t}}{16} \]

**Example 9.2** Find the inverse Laplace transform of \( F(s) = \frac{s}{(s^2 + 1)^2} \).

If we express \( F \) as \( F(s) = \frac{s}{(s^2 + 1)} \cdot \frac{1}{(s^2 + 1)} = \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\} \),

we will see that \( f(t) = \mathcal{L}^{-1}\{F\} = \cos t * \sin t \).

\[ f(t) = \int_0^t \cos(x) \sin(t-x) \, dx = \frac{1}{2} \int_0^t [\sin(t-x+x) + \sin(t-x-x)] \, dx = \frac{1}{2} \int_0^t [\sin(t) + \sin(t-2x)] \, dx = \frac{1}{2} \left[ x \sin t + \frac{\cos(t-2x)}{2} \right]_0^t = \frac{1}{2} \left[ t \sin t + \frac{1}{2} (\cos t - \cos t) \right] = \frac{1}{2} t \sin t \]
9.2 Unit Step Function

The Heaviside step function (or unit step function) is defined as

\[ u_a(t) = u(t - a) = \begin{cases} 
0 & \text{if } t < a \\
1 & \text{if } t \geq a 
\end{cases} \] (9.4)

This is a simple on off function. It is especially useful to express discontinuous inputs.

Figure 9.1: \( u(t - a) \) and its effect on \( f(t) \)

**Theorem 9.2:** \([t–shifting]\) Let \( \mathcal{L}\{f(t)\} = F(s) \), then

\[ \mathcal{L}\{f(t - a) u(t - a)\} = e^{-as}F(s) \] (9.5)

**Proof:** Using the definition,

\[
\mathcal{L}\{f(t - a) u(t - a)\} = \int_0^\infty e^{-st} f(t - a) u(t - a) \, dt \\
= \int_0^\infty e^{-st} f(t - a) \, dt \\
= \int_0^\infty e^{-sa - sx} f(x) \, dx \quad (\text{where } x = t - a) \\
= e^{-as}F(s)
\]

**Example 9.3** Find the Laplace transform of \( g(t) = \begin{cases} 
0 & \text{if } t < 5 \\
t & \text{if } t \geq 5 
\end{cases} \)

We can express \( g(t) \) as \( g(t) = u(t - 5)f(t - 5) \) where \( f(t) = (t + 5) \). Then

\[ F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} + \frac{5}{s} \implies \mathcal{L}\{g(t)\} = e^{-5s} \left( \frac{1}{s^2} + \frac{5}{s} \right) \]

9.3 Differentiation of Transforms

If \( f(t) \) is piecewise continuous and of exponential order, then we can differentiate its Laplace transform integral.
\[ F(s) = \int_0^\infty e^{-st} f(t) \, dt \]
\[ F'(s) = \int_0^\infty (-t) e^{-st} f(t) \, dt \]  
\[ (9.6) \]

In other words
\[ \mathcal{L}\{tf(t)\} = -F'(s) \]  
\[ (9.7) \]

Repeating this procedure \( n \) times, we obtain:
\[ \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \]  
\[ (9.8) \]

**Example 9.4** Find the Laplace transform of \( f(t) = t \sin t \).

Using the derivative formula, we find
\[ \mathcal{L}\{t \sin t\} = -\frac{d}{ds} \left( \frac{1}{1 + s^2} \right) = \frac{2s}{(1 + s^2)^2} \]
9.4 Partial Fractions Expansion

In many applications of Laplace transform, we need to expand a rational function in partial fractions. Here, we will review this technique by examples.

\[
\begin{align*}
\frac{2x + 1}{(x - 2)(x + 3)(x - 1)} &= \frac{A}{x - 2} + \frac{B}{x + 3} + \frac{C}{x - 1} \\
\frac{x^2 + 4x - 5}{(x - 2)(x - 1)^3} &= \frac{A}{x - 2} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3} \\
\frac{x^3 + 1}{x(x^2 + 4)^2} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2} \\
\frac{x^3 - 4x^2 + x + 9}{x^2 - 5x + 6} &= x + 1 + \frac{3}{x^2 - 5x + 6} = x + 1 + \frac{A}{x - 2} + \frac{B}{x - 3}
\end{align*}
\]

- We can express any polynomial as a product of first and second order polynomials.
- For second order polynomials in the expansion, we have to use \( Ax + B \) (not simply a constant) in the numerator.
- If numerator’s degree is greater or equal to the denominator, we should first divide them using polynomial division.

**Example 9.5** Find the inverse Laplace transform of \( F(s) = \frac{-s^2 + 7s - 1}{(s - 2)(s - 5)^2} \).

First, we have to express \( F(s) \) in terms of simpler fractions:

\[
\frac{-s^2 + 7s - 1}{(s - 2)(s - 5)^2} = \frac{A}{s - 2} + \frac{B}{s - 5} + \frac{C}{(s - 5)^2}
\]

\(-s^2 + 7s - 1 = A(s - 5)^2 + B(s - 2)(s - 5) + C(s - 2)\)

Inserting \( s = 2 \), we see that \( 9 = 9A \Rightarrow A = 1 \).

Inserting \( s = 5 \), we see that \( 9 = 3C \Rightarrow C = 3 \).

The coefficient of \( s^2 \): \( A + B = -1 \) therefore \( B = -2 \). So

\[
\frac{-s^2 + 7s - 1}{(s - 2)(s - 5)^2} = \frac{1}{s - 2} - \frac{2}{s - 5} + \frac{3}{(s - 5)^2}
\]

Now we can easily find the inverse Laplace transform:

\[
\mathcal{L}^{-1}\{F(s)\} = e^{2t} - 2e^{5t} + 3te^{5t}
\]
9.5 Applications

Now we are in a position to solve a wider class of differential equations using Laplace transform.

Example 9.6 Solve the initial value problem

\[ y'' - 6y' + 8y = 2e^{2t}, \quad y(0) = 11, \quad y'(0) = 37 \]

We will first find the Laplace transform of both sides, then find \( Y(s) \)

\[
\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \mathcal{L}\{2e^{2t}\}
\]

\[
s^2Y - 11s - 37 - 6(sY - 11) + 8Y = \frac{2}{s - 2}
\]

\[
(s^2 - 6s + 8)Y = \frac{2}{s - 2} + 11s - 29
\]

The factors of \( s^2 - 6s + 8 \) are \((s - 2)\) and \((s - 4)\), so

\[
Y = \frac{2}{(s - 2)(s - 2)(s - 4)} + \frac{11s - 29}{(s - 2)(s - 4)}
\]

\[
Y = \frac{11s^2 - 51s + 60}{(s - 2)^2(s - 4)}
\]

Now we need to find the inverse Laplace transform. Using partial fractions expansion

\[
Y = \frac{A}{s - 2} + \frac{B}{(s - 2)^2} + \frac{C}{s - 4}
\]

After some algebra we find that \( A = 3, \quad B = -1, \quad C = 8 \) so

\[
Y(s) = \frac{3}{s - 2} - \frac{1}{(s - 2)^2} + \frac{8}{s - 4}
\]

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} = 3e^{2t} - te^{2t} + 8e^{4t}
\]
Example 9.7 Solve the initial value problem

\[ y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 3 \]

where \( f(t) = \begin{cases} 
0 & \text{if } 0 < t < 5\pi \\
2 \cos t & \text{if } 5\pi < t 
\end{cases} \)

As you can see, the input function is discontinuous, but this makes no difference for Laplace transform.

\[
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f\}
\]

\[
s^2Y - 3 + Y = F
\]

\[
Y = \frac{F + 3}{s^2 + 1}
\]

Using the fact that \( \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} \), we can obtain \( y(t) \) by convolution:

\[
y(t) = \mathcal{L}^{-1}\{Y\} = f(t) * \sin t + 3 \sin t
\]

Using the definition of convolution,

\[
f * \sin t = \int_{0}^{t} f(x) \sin(t - x) \, dx
\]

If \( t < 5\pi \), \( f = 0 \) therefore this integral is also zero. If \( t > 5\pi \) we have

\[
f * \sin t = \int_{5\pi}^{t} 2 \cos x \sin(t - x) \, dx
\]

Using the trigonometric identity \( 2 \sin A \cos B = \sin(A + B) + \sin(A - B) \) we obtain

\[
f * \sin t = \int_{5\pi}^{t} \sin t + \sin(t - 2x) \, dx
\]

\[
= \left[ x \sin t + \frac{\cos(t - 2x)}{2} \right]_{5\pi}^{t}
\]

Therefore

\[
y(t) = \begin{cases} 
3 \sin t & \text{if } 0 < t < 5\pi \\
(t - 5\pi + 3) \sin t & \text{if } 5\pi < t
\end{cases}
\]
Example 9.8 Solve the initial value problem

\[ y'' + 2y' + y = r(t), \quad y(0) = 0, \quad y'(0) = 0 \]

where \( r(t) = \begin{cases} 
  t & \text{if } 0 < t < 1 \\
  0 & \text{if } 1 < t 
\end{cases} \)

Once again we have a discontinuous input. This time we will use unit step function. First, we have to express \( r(t) \) with a single formula.

\[ r(t) = t - u(t-1) = t - u(t-1)(t-1) - u(t-1) \]

Its Laplace transform is

\[ R(s) = \mathcal{L}\{r(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \]

Finding the Laplace transform of the equation, we obtain

\[ (s^2 + 2s + 1)Y = R \]

\[ Y = \frac{R}{(s + 1)^2} \]

\[ Y = \frac{1}{s^2(s + 1)^2} - \frac{e^{-s}}{s^2(s + 1)} \]

Using partial fractions expansion

\[ Y = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s + 1} + \frac{1}{(s + 1)^2} - e^{-s} \left( -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s + 1} \right) \]

Using the fact that \( \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \), we obtain

\[ y(t) = -2 + t + 2e^{-t} + te^{-t} - u(t-1) \left( -1 + (t-1) + e^{-(t-1)} \right) \]

We know that \( u(t-1) = 0 \) for \( t > 1 \) and \( u(t-1) = 1 \) for \( t > 1 \) so

\[ y(t) = \begin{cases} 
  -2 + t + 2e^{-t} + te^{-t} & \text{if } 0 < t < 1 \\
  (2 - e)e^{-t} + te^{-t} & \text{if } 1 < t 
\end{cases} \]
Exercises

Find the Laplace transform of the following functions:
1) \( f(t) = te^{-t} \cos t \)
2) \( f(t) = t^2 \sin 2t \)

Find the inverse Laplace transform of the following functions:
3) \( F(s) = \frac{e^{-3s}}{s^2 + 1} \)
4) \( F(s) = \frac{se^{-s}}{s^2 + 4} \)
5) \( F(s) = \frac{1}{(s^2 + 16)^2} \)
6) \( F(s) = \frac{1}{s^3 + 4s^2 + 3s} \)
7) \( F(s) = \frac{s + 3}{(s^2 + 4)^2} \)
8) \( F(s) = \frac{s^3}{s^4 + 4a^4} \)
9) \( F(s) = \frac{s^2}{(s^2 + 4)^2} \)
10) \( F(s) = \frac{3s^2 - 2s + 5}{(s^2 + 9)(s - 2)} \)

Solve the following initial value problems: (where \( y = y(t) \))
11) \( y'' - y' - 2y = 0, \ y(0) = 8, \ y'(0) = 7 \)
12) \( y'' + y = 2 \cos t, \ y(0) = 3, \ y'(0) = 4 \)
13) \( y'' + 0.64y = 5.12t^2, \ y(0) = -25, \ y'(0) = 0 \)
14) \( y'' - 2y' + 2y = e^{-t}, \ y(0) = 0, \ y'(0) = 1 \)
15) \( y'' + y = t, \ y(0) = 0, \ y'(0) = 0 \)
16) \( y'' + y = r(t), \ y(0) = 1, \ y'(0) = 0 \) where \( r(t) = \begin{cases} 1 & \text{if } 0 < t < 2\pi \\ 0 & \text{if } 2\pi < t \end{cases} \)
17) \( y'' + y = e^{-2t} \sin t, \ y(0) = 0, \ y'(0) = 0 \)
18) \( y'' + 2y' + 5y = r(t), \ y(0) = 0, \ y'(0) = 0 \) where \( r(t) = \begin{cases} 5 & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t \end{cases} \)
19) \( 4y'' + 4y' + 17y = g(t), \ y(0) = 0, \ y'(0) = 0 \)
20) \( y'' - y' - 6y = \begin{cases} \sin t & \text{if } 0 < t < 3\pi \\ 0 & \text{if } 3\pi < t \end{cases}, \ y(0) = \frac{1}{50}, \ y'(0) = -\frac{7}{50} \)
Answers

1) \( F(s) = \frac{s^2 + 2s}{(s^2 + 2s + 2)^2} \)

2) \( F(s) = \frac{12s^2 - 16}{(s^2 + 4)^3} \)

3) \( f(t) = u(t - 3) \sin(t - 3) \)

4) \( f(t) = u(t - 1) \cos(2t - 2) \)

5) \( f(t) = \frac{\sin 4t - 4t \cos 4t}{128} \)

6) \( f(t) = \frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \)

7) \( f(t) = \frac{4t \sin 2t + 3 \sin 2t - 6t \cos 2t}{16} \)

8) \( f(t) = \cosh at \cos at \)

9) \( f(t) = \frac{1}{4} \sin 2t + \frac{t}{2} \cos 2t \)

10) \( f(t) = e^{2t} + 2 \cos 3t + \frac{2}{3} \sin 3t \)

11) \( y = 3e^{-t} + 5e^{2t} \)

12) \( y = 3 \cos t + (4 + t) \sin t \)

13) \( y = -25 + 8t^2 \)

14) \( y = \frac{1}{5} (e^{-t} - e^t \cos t + 7e^t \sin t) \)

15) \( y = t - \sin t \)

16) \( y = \begin{cases} 1 & 0 < t < 2\pi \\ \cos t & 2\pi < t \end{cases} \)

17) \( y = \frac{1}{8} (\sin t - \cos t) + \frac{1}{8} e^{-2t} (\sin t + \cos t) \)

18) \( y = \begin{cases} 1 - e^{-t} \left( \cos 2t + \frac{\sin 2t}{2} \right) & 0 < t < \pi \\ e^{-t} (e^{\pi} - 1) \left( \cos 2t + \frac{\sin 2t}{2} \right) & \pi < t \end{cases} \)

19) \( y = \frac{1}{8} \int_0^t e^{-\frac{1}{2}(t-x)} \sin 2(t-x)g(x) \, dx \)

20) \( y = \begin{cases} \frac{1}{50} (\cos t - 7 \sin t) & \text{if } 0 < t < 3\pi \\ \frac{1}{50} e^{-9\pi} e^{3t} - \frac{2}{50} e^{6\pi} e^{-2t} & \text{if } 3\pi < t \end{cases} \)
Chapter 10

Fourier Analysis I

The trigonometric functions *sine* and *cosine* are the simplest periodic functions. If we can express an arbitrary periodic function in terms of these, many problems would be simplified. In this chapter, we will see how to find the Fourier series of a periodic function. Fourier series is important in many applications. We will also need them when we solve partial differential equations.

10.1 Fourier Series

Let \( f(x) \) be a periodic function with period \( 2L \). It is sufficient that \( f \) be defined on \([-L, L]\). Is it possible to express \( f \) as a linear combination of sine and cosine functions?

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{10.1}
\]

If possible, this expansion would be very useful in all kinds of applications. Once we solve a question for sine and cosine functions, we will be able to solve it for any periodic \( f \). Here, \( a_n \) and \( b_n \) are the coordinates of \( f \) in the space of sine and cosine functions. But then how can we find \( a_n \) and \( b_n \)? The following identities will help us:

\[
\int_{-L}^{L} \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = 0 \text{ (for all } m, n \text{)} \tag{10.2}
\]
\[
\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0 \quad (m \neq n) \quad (10.3)
\]
\[
\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = 0 \quad (m \neq n) \quad (10.4)
\]
\[
\int_{-L}^{L} \cos^{2} \frac{n\pi x}{L} \, dx = \int_{-L}^{L} \sin^{2} \frac{n\pi x}{L} \, dx = L \quad (10.5)
\]

In the terminology of linear algebra, the trigonometric functions form an orthogonal coordinate basis. We can easily prove these formulas if we remember the following trigonometric identities:

\[
2 \cos A \cos B = \cos(A - B) + \cos(A + B)
\]

\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]

\[
2 \cos A \sin B = \sin(A + B) - \sin(A - B)
\]

(10.6)

Now, suppose the expansion (10.1) exists. To find \(a_k\), we will multiply both sides by \(\cos \frac{k\pi x}{L}\) and then integrate from \(-L\) to \(L\).

\[
\int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} \, dx = \int_{-L}^{L} a_0 \cos \frac{k\pi x}{L} \, dx \\
+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{k\pi x}{L} \, dx \\
+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{k\pi x}{L} \, dx
\]

(10.7)

Using the property of orthogonality, we can see that all those integrals are zero, except the \(k^{th}\) one. Therefore

\[
\int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} \, dx = a_k L \quad \Rightarrow \quad a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} \, dx
\]

(10.8)

We can apply the same procedure to find \(a_0\) and \(b_n\). In the end, we will obtain the following formulas for a function \(f\) defined on \([-L, L]\).
Fourier coefficients:

\[
\begin{align*}
    a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
    a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \\
    b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\end{align*}
\] (10.9)

Fourier series:

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)
\] (10.10)

**Example 10.1** Find the Fourier series of the periodic function \( f(x) = x^2, \quad -L \leq x \leq L \) having period \( 2L \).

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} x^2 \, dx = \frac{1}{2L} \left[ \frac{x^3}{3} \right]_{-L}^{L} = \frac{L^2}{3}
\]

Using integration by parts two times we find:

\[
a_n = \frac{1}{L} \int_{-L}^{L} x^2 \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{4L^2 \cos n\pi}{n^2\pi^2}
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} x^2 \sin \left( \frac{n\pi x}{L} \right) \, dx = 0
\]

Therefore the Fourier series is:

\[
x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4L^2}{n^2\pi^2} \cos \left( \frac{n\pi x}{L} \right)
\]

The plot of the Fourier series up to \( n = 1, 2 \) and 3 is given in Figure 10.1.
Figure 10.1: Fourier Series of $f = x^2$ for $n = 1, 2, 3$
10.2 Convergence of Fourier Series

Like any infinite series, Fourier series is of no use if it is divergent. But most functions that we are interested in have Fourier series that converge and converge to the function.

**Theorem 10.1**: Let \( f \) be periodic with period \( 2L \) and let \( f \) and \( f' \) be piecewise continuous on the interval \([−L, L]\). Then the Fourier expansion of \( f \) converges to:

- \( f(x) \) if \( f \) is continuous at \( x \).
- \( \frac{f(x^+) + f(x^-)}{2} \) if \( f \) is discontinuous at \( x \).

**Example 10.2** Find the Fourier series of the periodic function
\[
f(x) = \begin{cases} 
a & \text{if } -L < x < 0 \\
b & \text{if } 0 < x < L \\end{cases}
\]

having period \( 2L \). Then evaluate the series at \( x = L \).

\[
a_0 = \frac{1}{2L} \int_{-L}^{0} a \, dx + \frac{1}{2L} \int_{0}^{L} b \, dx = \frac{a + b}{2}
\]

\[
a_n = \frac{1}{L} \int_{-L}^{0} a \cos \frac{n\pi x}{L} \, dx + \frac{1}{L} \int_{0}^{L} b \cos \frac{n\pi x}{L} \, dx = 0
\]

\[
b_n = \frac{1}{L} \int_{-L}^{0} a \sin \frac{n\pi x}{L} \, dx + \frac{1}{L} \int_{0}^{L} b \sin \frac{n\pi x}{L} \, dx
\]

\[
= -\frac{a}{L} \left. \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right|_{-L}^{0} - \frac{b}{L} \left. \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right|_{0}^{L} = \frac{b - a}{n\pi} (1 - (-1)^n)
\]

Therefore the Fourier series is:
\[
f(x) = \frac{a + b}{2} + \sum_{n=1}^{\infty} \frac{b - a}{n\pi} \left[1 - (-1)^n\right] \sin \frac{n\pi x}{L}
\]

\[
= \frac{a + b}{2} + \frac{2(b - a)}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right)
\]

If we insert \( x = L \) in that series, we obtain \( f(L) = \frac{a + b}{2} \). Thus the value at discontinuity is the average of left and right limits. The summation of the series up to \( n = 1, 5 \) and 9 is plotted on Figure 10.2.
10.3 Parseval’s Identity

**Theorem 10.2:** Let \( f \) be continuous on \([-L, L]\), \( f(L) = f(-L) \) and let \( f' \) be piecewise continuous. Then the Fourier coefficients of \( f \) satisfy:

\[
2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx
\]  
(10.11)

**Proof:** We can express \( f(x) \) as

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.
\]

Now multiply both sides by \( f(x) \) and integrate

\[
\int_{-L}^{L} f^2(x) \, dx = a_0 \int_{-L}^{L} f(x) \, dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]

Using equation (10.9) to evaluate these integrals, we can obtain the result.

**Example 10.3** Find the sum of the series

\[
S = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots
\]

(Hint: Use the Fourier series of \( f(x) = x^2 \) on the interval \(-\pi < x < \pi\))

Evaluating the integrals in (10.9) for \( f(x) = x^2 \) we obtain

\[
a_0 = \frac{\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2} \text{ and } b_n = 0
\]

so

\[
f(x) = \frac{\pi^2}{3} - 4 \left( \cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \cdots \right)
\]

Using Parseval’s theorem, we have

\[
\frac{2\pi^4}{9} + 16 \left( 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx
\]

\[
= \frac{2}{5} \pi^4
\]

Therefore
\[ 16 \left( 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \right) = \pi^4 \left( \frac{2}{5} - \frac{2}{9} \right) \]

\[ S = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \]
Exercises

Find the Fourier series of the periodic function \( f(x) \) defined on the given interval

1) \( f(x) = x, \ -\pi < x < \pi \)

2) \( f(x) = x, \ 0 < x < 2\pi \)

3) \( f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases} \)

4) \( f(x) = x^2, \ 0 < x < 2\pi \)

5) \( f(x) = \sin^2 x, \ -\pi < x < \pi \)

6) \( f(x) = x + |x|, \ -\pi < x < \pi \)

7) \( f(x) = \begin{cases} -\pi/4 & \text{if } -1 < x < 0 \\ \pi/4 & \text{if } 0 < x < 1 \end{cases} \)

8) \( f(x) = \begin{cases} \pi & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi \end{cases} \)

9) \( f(x) = |x|, \ -2 < x < 2 \)

10) \( f(x) = |\sin x|, \ -\pi < x < \pi \)

11) \( f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 - x & \text{if } 1 < x < 2 \end{cases} \)

12) \( f(x) = \begin{cases} -a & \text{if } -L < x < 0 \\ a & \text{if } 0 < x < L \end{cases} \)

13) \( f(x) = ax + b, \ -L < x < L \)

14) \( f(x) = 1 - x^2, \ -1 < x < 1 \)

15) \( f(x) = x^3, \ -\pi < x < \pi \)

16) \( f(x) = e^x, \ -\pi < x < \pi \)

17) Using integration by parts, show that:

\[
\int x \cos ax \, dx = \frac{x \sin ax}{a} + \frac{\cos ax}{a^2}
\]

\[
\int x \sin ax \, dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2}
\]

\[
\int x^2 \cos ax \, dx = \frac{x^2 \sin ax}{a} + \frac{2x \cos ax}{a^2} - \frac{2 \sin ax}{a^3}
\]

\[
\int x^2 \sin ax \, dx = -\frac{x^2 \cos ax}{a} + \frac{2x \sin ax}{a^2} - \frac{2 \cos ax}{a^3}
\]

18) Show that \( 1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8} \).
EXERCISES

Answers

1) \( f(x) = 2\pi \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \cdots \right) \)

2) \( f(x) = \pi - 2 \left( \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \right) \)

3) \( f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin nx = \frac{1}{2} + \frac{2}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right) \)

4) \( f(x) = \frac{4\pi^2}{3} + 4 \left( \cos x + \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x + \cdots \right) \)

\[-4\pi \left( \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \right) \]

5) \( f(x) = \frac{1}{2} - \frac{1}{2} \cos 2x \)

6) \( f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \cdots \right) \)

\[+2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{4}{9} \sin 4x + \cdots \right) \]

7) \( f(x) = \sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x + \cdots \)

8) \( f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \left[ \frac{(-1)^n - 1}{\pi n^2} \cos nx - \frac{1}{n} \sin nx \right] \)

9) \( f(x) = 1 - \frac{8}{\pi^2} \left( \cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \cdots \right) \)

10) \( f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \)

11) \( f(x) = -\frac{4}{\pi^2} \left( \cos \pi x + \frac{1}{9} \cos 3\pi x + \frac{1}{25} \cos 5\pi x + \cdots \right) \)

\[+\frac{2}{\pi} \left( \sin \pi x + \frac{1}{3} \sin 3\pi x + \cdots \right) \)
12) $f(x) = \frac{4a}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right)$

13) $f(x) = b + \frac{2aL}{\pi} \left( \sin \frac{\pi x}{L} - \frac{1}{2} \sin \frac{2\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} - \cdots \right)$

14) $f(x) = \frac{2}{3} + \frac{4}{\pi^2} \left( \cos \pi x - \frac{1}{4} \cos 2\pi x + \frac{1}{9} \cos 3\pi x + \cdots \right)$

15) $f(x) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{(n\pi)^2 - 6}{n^3} \right] \sin nx$

16) $f(x) = \frac{2 \sinh \pi}{\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2} (\cos nx - n \sin nx) \right]$

18) Use the function in exercise 12 in Parseval’s identity
Chapter 11

Fourier Analysis II

In this chapter, we will study more advanced properties of Fourier series. We will find the even and odd periodic extensions of a given function, we will express the series using complex notation and finally, we will extend the idea of Fourier series to nonperiodic functions in the form of a Fourier integral.

11.1 Fourier Cosine and Sine Series

If \( f(−x) = f(x) \), \( f \) is an even function. If \( f(−x) = −f(x) \), \( f \) is an odd function. We can easily see that, for functions:

\[
\begin{align*}
\text{even} \times \text{even} &= \text{even}, \\
\text{odd} \times \text{odd} &= \text{even}, \\
\text{even} \times \text{odd} &= \text{odd}
\end{align*}
\]

For example \(|x|, x^2, x^4, \cos x, \cos nx, \cosh x\) are even functions. \( x, x^3, \sin x, \sin nx, \sinh x \) are odd functions. \( e^x \) is neither even nor odd.

If \( f \) is even:

\[
\int_{−L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx
\]  
\tag{11.1}

If \( f \) is odd:

\[
\int_{−L}^{L} f(x) \, dx = 0
\]  
\tag{11.2}

Using the above equations, we can see that in the Fourier expansion of an even function, \( b_n = 0 \), and in the expansion of an odd function, \( a_n = 0 \). This will cut our work in half if we can recognize the given function as odd or even.
As you can see in Figure 11.1, an even function is symmetric with respect to $y$-axis, an odd function is symmetric with respect to origin.

**Half Range Extensions:** Let $f$ be a function defined on $[0, L]$. If we want to expand it in terms of *sine* and *cosine* functions, we can think of it as periodic with period $2L$. Now we need to define $f$ on the interval $[-L, 0]$. There are infinitely many possibilities, but for simplicity, we are interested in making $f$ an even or an odd function. If we define $f$ for negative $x$ values as $f(x) = f(-x)$, we obtain the even periodic extension of $f$, which is represented by a Fourier cosine series. If we define $f$ for negative $x$ values as $f(x) = -f(-x)$, we obtain the odd periodic extension of $f$, which is represented by a Fourier sine series.

**Half-Range Cosine Expansion:** (or Fourier cosine series)

$$ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad (0 < x < L) \quad (11.3) $$

where $a_0 = \frac{1}{L} \int_0^L f(x) \, dx$, $a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx \quad (11.4)$
11.1. FOURIER COSINE AND SINE SERIES

Half-Range Sine Expansion: (or Fourier sine series)

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad (0 < x < L) \]  
(11.5)

where

\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \]  
(11.6)

Example 11.1 Find the half-range cosine and sine expansions of

\[ f(x) = \begin{cases} 
0 & \text{if } 0 < x < \frac{\pi}{2} \\
\frac{\pi}{2} & \text{if } \frac{\pi}{2} < x < \pi 
\end{cases} \]

Here, \( L = \pi \), therefore

\[
\begin{align*}
a_0 &= \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2} \, dx = \frac{\pi}{4} \\
a_n &= \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2} \cos nx \, dx \\
&= \left. \frac{\sin nx}{n} \right|_{\frac{\pi}{2}}^{\pi} = -\frac{\sin \frac{n\pi}{2}}{n}
\end{align*}
\]

Therefore half-range cosine series of \( f \) is

\[ f(x) = \frac{\pi}{4} - \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos nx = \frac{\pi}{4} - \left( \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right) \]

On the other hand,

\[
\begin{align*}
b_n &= \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2} \sin nx \, dx \\
&= \left. -\cos nx \right|_{\frac{\pi}{2}}^{\pi} = \frac{\cos \frac{n\pi}{2} - \cos n\pi}{n}
\end{align*}
\]

Therefore half-range sine series of \( f \) is

\[ f(x) = \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{2} - \cos n\pi}{n} \sin nx = \sin x - \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \]
11.2 Complex Fourier Series

Consider the Fourier series of $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$  \hspace{1cm} (11.7)

Using Euler’s formula $e^{ix} = \cos x + i \sin x$ we can express the sine and cosine functions as:

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$  \hspace{1cm} (11.8)

Therefore

$$a_n \cos nx + b_n \sin nx = \left( \frac{a_n - ib_n}{2} \right) e^{inx} + \left( \frac{a_n + ib_n}{2} \right) e^{-inx}$$  \hspace{1cm} (11.9)

If we define $c_0 = a_0$ and

$$c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n = 1, 2, 3, \ldots$$  \hspace{1cm} (11.10)

We will obtain

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$  \hspace{1cm} (11.11)

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad n = 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (11.12)

For a function of period $2L$ we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-inx/L} \, dx$$  \hspace{1cm} (11.13)

**Example 11.2** Find the complex Fourier series of

$$f(x) = x \text{ if } -\pi < x < \pi \text{ and } f(x + 2\pi) = f(x).$$

We have to evaluate the integral

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-inx} \, dx$$
For $n = 0$ this integral is zero, so we have $c_0 = 0$. For $n \neq 0$
\[
c_n = \frac{1}{2\pi} \left( \frac{xe^{-inx}}{-in} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx} \, dx \right) \\
= \frac{1}{2\pi} \left( \frac{\pi e^{-in\pi} + \pi e^{in\pi} - 0}{-in} \right) \\
= -\frac{1}{in} \frac{e^{in\pi} + e^{-in\pi}}{2} = -\frac{\cos n\pi}{in} \\
= \frac{i}{n} (-1)^n
\]

Therefore
\[
x = \sum_{n=-\infty}^{\infty} \frac{i}{n} (-1)^n e^{inx}, \quad n \neq 0
\]

Note that we can obtain the real Fourier series from the complex one. If we add $n^{th}$ and $-n^{th}$ terms we get
\[
i(-1)^n \frac{\cos nx + i \sin nx}{n} + i(-1)^{-n} \frac{\cos(-nx) + i \sin(-nx)}{-n} = (-1)^{n+1} \frac{\sin nx}{n}
\]

This is the real Fourier series.

**Example 11.3** Find the complex Fourier series of $f(x) = k$

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} ke^{-inx} \, dx \\
= \frac{k}{2\pi} \left( \frac{e^{-inx}}{-in} \bigg|_{-\pi}^{\pi} \right) (n \neq 0) \\
= \frac{k}{n\pi} \frac{e^{in\pi} - e^{-in\pi}}{2i} \\
= \frac{k}{n\pi} \sin n\pi \\
= 0
\]

If $n = 0$ we have
\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} k \, dx \\
= k
\]
11.3 Fourier Integral Representation

In this section, we will apply the basic idea of the Fourier series to non-periodic functions.

Consider a periodic function with period $2L$ and its Fourier series. In the limit $L \to \infty$, the summation will be an integral, and $f$ will be a non-periodic function. Then we will obtain the Fourier integral representation:

$$f(x) = \int_0^\infty [A(u) \cos u x + B(u) \sin u x] \, du$$  \hspace{1cm} (11.14)

where

$$A(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos u x \, dx$$  \hspace{1cm} (11.15)

$$B(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin u x \, dx$$  \hspace{1cm} (11.16)

Like the Fourier series, we have $A(u) = 0$ for odd functions and $B(u) = 0$ for even functions.

**Theorem 11.1**: If $f$ and $f'$ are piecewise continuous in every finite interval and if $\int_{-\infty}^{\infty} |f| \, dx$ is convergent, then the Fourier integral of $f$ converges to:

- $f(x)$ if $f$ is continuous at $x$.
- $\frac{f(x+) + f(x-)}{2}$ if $f$ is discontinuous at $x$.

**Example 11.4** Find the Fourier integral representation of

$$f(x) = \begin{cases} \pi/2 & \text{if } |x| < 1 \\ 0 & \text{if } 1 < |x| \end{cases}$$

Note that $f$ is even therefore $B(u) = 0$

$$A(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos u x \, dx = \frac{1}{\pi} \int_{-1}^{1} \frac{\pi}{2} \cos u x \, dx$$

$$= \int_{0}^{1} \cos u x \, dx = \frac{\sin u x}{u} \bigg|_{0}^{1} = \frac{\sin u}{u}$$
Therefore, Fourier integral representation of $f$ is

$$f(x) = \int_{0}^{\infty} \frac{\sin u}{u} \cos ux \, du$$

**Example 11.5** Prove the following formulas using two different methods:

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

We can obtain the formulas using integration by parts, but this is the long way. A better method is to express the integrals as a single complex integral using $e^{ibx} = \cos bx + i \sin bx$, then evaluate it at one step, and then separate the real and imaginary parts.

**Example 11.6** Find the Fourier integral representation of

$$f(x) = \begin{cases} 
-e^x \cos x & \text{if } x < 0 \\
e^{-x} \cos x & \text{if } 0 < x 
\end{cases}$$

This function is odd therefore $A(u) = 0$.

$$B(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin ux \, dx = \frac{2}{\pi} \int_{0}^{\infty} e^{-x} \cos x \sin ux \, dx$$

$$= \frac{2}{\pi} \left[ -e^{-x} \frac{\sin(ux + x) + \sin(ux - x)}{2} \right]_{0}^{\infty}$$

$$= \frac{1}{\pi} \left. \frac{e^{-x}}{1 + (u + 1)^2} [-\sin(u + 1)x - (u + 1) \cos(u + 1)x] \right|_{0}^{\infty}$$

$$+ \frac{1}{\pi} \left. \frac{e^{-x}}{1 + (u - 1)^2} [-\sin(u - 1)x - (u - 1) \cos(u - 1)x] \right|_{0}^{\infty}$$

$$= \frac{1}{\pi} \left( \frac{u + 1}{1 + (u + 1)^2} + \frac{u - 1}{1 + (u - 1)^2} \right)$$

$$= \frac{2}{\pi} \frac{u^3}{u^4 + 4}$$

So

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{u^3}{u^4 + 4} \sin ux \, du$$
Exercises

For the following functions defined on $0 < x < L$, find the half-range cosine and half-range sine expansions:

1) $f(x) = \begin{cases} \frac{2kx}{L} & \text{if } 0 < x < L/2 \\ \frac{2k(L-x)}{L} & \text{if } L/2 < x < L \end{cases}$

2) $f(x) = e^x$

3) $f(x) = k$

4) $f(x) = x^4$

5) $f(x) = \cos 2x$ $0 < x < \pi$

6) $f(x) = \begin{cases} 0 & \text{if } 0 < x < L/2 \\ k & \text{if } L/2 < x < L \end{cases}$

Find the complex Fourier series of the following functions:

7) $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

8) $f(x) = x^2$, $-L < x < L$

9) $f(x) = \sin x$

10) $f(x) = \cos 2x$

Find the Fourier integral representations of the following functions:

11) $f(x) = \begin{cases} \pi - x & 0 < x < \pi \\ 0 & \pi < x \end{cases}$ (f odd)

12) $f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$

13) $f(x) = \begin{cases} e^{-x}, & 0 < x \\ e^x, & x < 0 \end{cases}$

14) $f(x) = \begin{cases} \pi & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$

Prove the following formulas. (Hint: Define a suitable function $f$ and then find its Fourier integral representation.)

15) $\int_0^\infty \left[ \left(1 - \frac{2}{u^2}\right) \sin u + \frac{2}{u} \cos u \right] \frac{\cos ux}{u} \, du = \begin{cases} \frac{\pi x^2}{2}, & 0 \leq x < 1 \\ \pi/4, & x = 1 \\ 0, & 1 < x \end{cases}$

16) $\int_0^\infty \frac{\cos ux + u \sin ux}{1 + u^2} \, du = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$
Answers

1) \[ f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left( \frac{1}{2^2} \cos \frac{2\pi x}{L} + \frac{1}{6^2} \cos \frac{6\pi x}{L} + \cdots \right) \]

\[ f(x) = \frac{8k}{\pi^2} \left( \frac{1}{12} \sin \frac{\pi x}{L} - \frac{1}{3^2} \sin \frac{3\pi x}{L} + \frac{1}{5^2} \sin \frac{5\pi x}{L} - \cdots \right) \]

2) \[ f(x) = \frac{1}{L} (e^L - 1) + \sum_{n=1}^{\infty} \frac{2L}{L^2 + n^2 \pi^2} [(-1)^n e^L - 1] \cos \frac{n\pi x}{L} \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{L^2 + n^2 \pi^2} [1 - (-1)^n e^L] \sin \frac{n\pi x}{L} \]

3) \[ f(x) = k \]

\[ f(x) = \frac{4k}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right) \]

4) \[ f(x) = \frac{L^4}{5} + 8L^4 \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n^2 \pi^2} - \frac{6}{n^4 \pi^4} \right) \cos \frac{n\pi x}{L} \]

\[ f(x) = 2L^4 \sum_{n=1}^{\infty} [(-1)^{n+1} \left( \frac{1}{n\pi} - \frac{12}{n^3 \pi^3} + \frac{24}{n^5 \pi^5} \right) + \frac{24}{n^5 \pi^5} ] \sin \frac{n\pi x}{L} \]

5) \[ f(x) = \cos 2x \]

\[ f(x) = -\frac{4}{3\pi} \sin x + \frac{2}{\pi} \sum_{n=3}^{\infty} [1 - (-1)^n] \frac{n}{n^2 - 4} \sin nx \]

6) \[ f(x) = \frac{k}{2} - \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{L}}{n} \cos \frac{n\pi x}{L} \]

\[ f(x) = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi x}{L} - \cos n\pi}{n} \sin \frac{n\pi x}{L} \]

7) \[ f(x) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{i}{2\pi n} [(-1)^n - 1] e^{inx}, \quad n \neq 0 \]

8) \[ f(x) = \frac{L^2}{3} + \frac{2L^2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2} e^{inx/L}, \quad n \neq 0 \]
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9) \( f(x) = -\frac{i}{2}e^{ix} + \frac{i}{2}e^{-ix} \)

10) \( f(x) = \frac{1}{2}e^{2ix} + \frac{1}{2}e^{-2ix} \)

11) \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\pi u - \sin \pi u}{u^2} \sin xu \, du \)

12) \( f(x) = \int_{0}^{\infty} \frac{\cos (\frac{\pi u}{2}) \cos xu}{1 - u^2} \, du \)

13) \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos xu}{1 + u^2} \, du \)

14) \( f(x) = \int_{0}^{\infty} \left[ \left( \frac{1 - \cos u}{u} \right) \sin ux + \frac{\sin u}{u} \cos ux \right] \, du \)
Chapter 12

Partial Differential Equations, Wave Equation

All the differential equations we have seen up to now were ordinary, that is, they had one independent variable. In real life, almost any problem has more than one independent variables. Therefore the subject of partial differential equations is vast and complicated. In this chapter we will see how to model a physical situation to set up an equation. We will obtain a solution using the method of separation of variables. Fourier series and ODE solutions will be necessary in this process.

12.1 Introduction

An equation involving partial derivatives of an unknown function is called a partial differential equation, or PDE for short. Mathematical formulation of problems where there are more than one independent variables require PDE’s and they are usually much more complicated than ODE’s. (Ordinary Differential Equations)

The definition of linear, nonlinear, homogeneous and nonhomogeneous equations are similar to that of ODE’s. So, a general second order linear partial differential equation is:

\[ \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G \quad (12.1) \]
CHAPTER 12. PARTIAL DIFFERENTIAL EQUATIONS

where the unknown function is $u$ and the two independent variables are $x$ and $y$. Here $A, B, \ldots, G$ are functions that may depend on $x$ and $y$ but not on $u$. If $G$ is zero, the equation is homogeneous, otherwise it is nonhomogeneous.

We can generalize these concepts into higher order PDE’s, but we will work with second order equations in the remainder of this book. A lot of problems in elastic vibrations, heat conduction, potential theory, wave propagation and quantum mechanics can be formulated by second order linear PDE’s.

**Examples:** All of the following are linear and homogeneous equations:

- Wave equation in one dimension
  \[ u_{tt} - c^2 u_{xx} = 0 \]  
  (12.2)

- Wave equation in three dimensions
  \[ u_{tt} - c^2 \nabla^2 u = 0 \]  
  (12.3)

- Heat equation in one dimension
  \[ u_t - \kappa u_{xx} = 0 \]  
  (12.4)

Laplace equation in Cartesian coordinates:
\[ \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \]  
(12.5)

Laplace equation in cylindrical coordinates: \( (x = \rho \cos \theta, \ y = \rho \sin \theta) \)
\[ u_{\rho\rho} + \frac{u_{\rho}}{\rho} + \frac{u_{\theta\theta}}{\rho^2} + u_{zz} = 0 \]  
(12.6)

**Solutions:** Many different functions may solve a given PDE, for example the functions

\[
\begin{align*}
  u(x, t) &= \cos ct \sin x \\
  u(x, t) &= 4e^{ct}e^{-x} \\
  u(x, t) &= (4x - 6)(10t + 1) \\
  u(x, t) &= (x - ct)^5
\end{align*}
\]  
(12.7)

are all solutions to equation 12.2. (Please verify.)

**Initial and Boundary Conditions:** If the unknown function is specified at a certain time, this is called an Initial Condition (IC). If it is specified at the boundary of a region, it is called a Boundary Condition (BC).

**Superposition of Solutions:** If $u_1$ and $u_2$ satisfy a linear homogeneous PDE, then a linear combination of them (i.e. $c_1 u_1 + c_2 u_2$) also satisfies the same equation.
12.2 Modeling a Vibrating String

Figure 12.1: A piece of a vibrating string

Consider a small part of a string with linear mass density $\rho$ and the length of the undeflected string $\Delta x$. (Figure 12.1) There’s no motion in the horizontal direction, so the net force must be zero in this direction:

$$T_1 \cos \theta_1 = T_2 \cos \theta_2 = T$$

(12.8)

Here $T$ denotes the horizontal component of tension. The net force is mass times acceleration by Newton’s second law, so

$$T_2 \sin \theta_2 - T_1 \sin \theta_1 = \rho \Delta x \, u_{tt}$$

$$T (\tan \theta_2 - \tan \theta_1) = \rho \Delta x \, u_{tt}$$

(12.9)

We know that $\tan \theta$ is the same thing as the value of the derivative at that point, therefore:

$$\frac{\partial u}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial u}{\partial x} \bigg|_{x} = \frac{\rho}{T} \, u_{tt}$$

(12.10)

In the limit $\Delta x \to 0$ the expression on the left becomes the second derivative at $x$. Using $c^2 = \frac{T}{\rho}$ we obtain the one-dimensional wave equation:

$$u_{tt} = c^2 \, u_{xx}$$

(12.11)

Here $c$ is the wave velocity. As you can see, the velocity depends on tension and linear density of the string.

12.3 Method of Separation of Variables

This is the basic method we will use in the solution of PDE’s. The idea is as follows:

- Assume that the solution $u(x, t)$ is $u(x, t) = F(x)G(t)$.
- Insert this in the equation. Transform the PDE into two ODE’s.
• Solve the ODE’s. Then, superpose all the solutions.
• Find the solutions that satisfy the given boundary and initial conditions

There are a lot of tricks and details in the process that are best explained on an example:

Example 12.1 Formulate and solve the problem of motion of a guitar string that is initially given a shape as seen in Figure 12.2 and no initial velocity.

Figure 12.2: The initial shape of a guitar string

We know that the PDE satisfied by a vibrating string is:

\[ u_{tt} = c^2 u_{xx} \]

The string is fixed at the points \( x = 0 \) and \( x = L \) therefore the Boundary Conditions are

\[ u(0, t) = 0, \quad u(L, t) = 0 \]

The initial displacement is given in the figure, and the initial velocity is zero, therefore

\[ u(x, 0) = \begin{cases} \frac{2h}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2h}{L}(L - x) & \text{if } \frac{L}{2} < x < L \end{cases} \]

\[ \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} = 0 \]

This is the typical formulation of a PDE together with BC and IC. Now we start the method of separation of variables by assuming \( u(x, t) = F(x)G(t) \), then

\[ u_{tt} = FG'', \quad u_{xx} = F''G \quad \Rightarrow \quad FG'' = c^2 F''G \]

\[ \frac{G''}{c^2 G} = \frac{F''}{F} = k \]

Note that the left hand side depends on \( t \) only and the right hand side depends on \( x \) only, so this equality is possible only if both are equal to a constant. Therefore

\[ \frac{G''}{c^2 G} = \frac{F''}{F} = k \]
12.3. METHOD OF SEPARATION OF VARIABLES

Case 1) $k > 0, k = p^2, F = Ae^{px} + Be^{-px}$, using the BC we find

$$A + B = 0, \quad Ae^{pL} + Be^{-pL} = 0$$

Inserting $B = -A$ in the second equation, we get

$$A(e^{pL} - e^{-pL}) = 0, \quad p \neq 0 \quad \Rightarrow \quad A = 0, B = 0$$

therefore $F = 0$ and $u = FG = 0$ so the solution is trivial.

Case 2) $k = 0, F'' = 0, F = Ax + B$, using the BC we find

$$B = 0, \quad AL + B = 0$$

therefore $A = 0$ and $F = 0$, $u = FG = 0$ so the solution is again trivial.

Case 3) $k < 0, k = -p^2, F = A\cos px + B\sin px$, using the BC we find

$$A = 0, \quad A\cos pL + B\sin pL = 0$$

Therefore $B\sin pL = 0$.

At this point, one possibility is to choose $B = 0$, but this would again give the trivial solution $u = 0$. An alternative is to make $\sin pL = 0$, which is possible if $pL = n\pi$. Therefore

$$p = \frac{n\pi}{L}, \quad (n = 1, 2, 3\ldots)$$

Now we have infinitely many different $F'$s, so let’s denote them by $F_n$.

$$F_n = B_n \sin \frac{n\pi x}{L}$$

$$G'' = -\frac{n^2\pi^2c^2}{L^2}G \quad \Rightarrow \quad G_n = K_n \cos \frac{n\pi ct}{L} + L_n \sin \frac{n\pi ct}{L}$$

The IC $\frac{\partial u(x,t)}{\partial t} \Bigg|_{t=0} = 0$ gives $L_n = 0$ so $u_n$ can be written as

$$u_n(x,t) = B_n K_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}$$

Without loss of generality, we can choose $K_n = 1$, because we do not need two arbitrary constants. Using the superposition principle, we have to add all the solutions to obtain the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}$$
The only condition we did not use is the IC $u(x,0) = f(x)$. This gives

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Therefore $B_n$ are the Fourier sine coefficients of $f(x)$. So

$$B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{L} \int_{0}^{L/2} 2hxL \sin \frac{n\pi x}{L} \, dx + \frac{2}{L} \int_{L/2}^{L} \left(2h - \frac{2hx}{L}\right) \sin \frac{n\pi x}{L} \, dx$$

Performing the integration, we find

$$B_n = \frac{8h}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

So the solution is

$$u(x,t) = \frac{8h}{\pi^2} \left( \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L} - \frac{1}{3^2} \sin \frac{3\pi x}{L} \cos \frac{3\pi ct}{L} + \cdots \right)$$

The plot of the solution $u(x,t)$ for selected times is given in Figure 12.3.

**Example 12.2** Solve the PDE $u_{tt} = c^2 u_{xx}$, with

- **BC:** $u(0, t) = u(L, t) = 0$
- **IC:** $u(x, 0) = 0$, $\frac{\partial u(x, 0)}{\partial t} = g(x)$

This question is very similar to the previous one, but this time initial deflection is zero and the initial velocity is nonzero.

Following the same steps as we did, we obtain

$$F_n = B_n \sin \frac{n\pi x}{L}$$

$$G'' = -\frac{n^2 \pi^2 c^2}{L^2} G$$
\[ G_n = K_n \cos \frac{n\pi ct}{L} + L_n \sin \frac{n\pi ct}{L} \]

The IC \( u(x, 0) = 0 \) gives \( K_n = 0 \) so \( u_n \) can be written as

\[ u_n(x, t) = B_n L_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L} \]

We choose \( L_n = 1 \) and superpose all the solutions to obtain

\[
\begin{align*}
  u(x, t) &= \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L} \\
\end{align*}
\]

The only condition we did not use is the IC

\[ \frac{\partial u(x, 0)}{\partial t} = g(x) \]

This gives

\[
\sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L} = g(x)
\]

Therefore \( \frac{n\pi c}{L} B_n \) are the Fourier sine coefficients of \( g(x) \), so

\[
B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx
\]
Exercises

1) Solve the PDE \( u_{tt} = 4u_{xx} \) on \( 0 < x < \pi, \ 0 < t \), with
   BC: \( u(0, t) = u(\pi, t) = 0 \)
   IC: \( u(x, 0) = \sin(2x), \quad \frac{\partial u(x, 0)}{\partial t} = 0 \)

2) Solve the PDE \( u_{tt} = u_{xx} \) on \( 0 < x < 1, \ 0 < t \), with
   BC: \( u(0, t) = u(1, t) = 0 \)
   IC: \( u(x, 0) = x(1 - x), \quad \frac{\partial u(x, 0)}{\partial t} = 0 \)

3) Solve the PDE \( u_{tt} = \frac{1}{9}u_{xx} \) on \( 0 < x < 2, \ 0 < t \), with
   BC: \( u(0, t) = u(2, t) = 0 \)
   IC: \( u(x, 0) = 5\sin(\pi x) - 3\sin(2\pi x), \quad \frac{\partial u(x, 0)}{\partial t} = 0 \)

4) Solve the PDE \( u_{tt} = c^2u_{xx} \) on \( 0 < x < L, \ 0 < t \), with
   BC: \( u(0, t) = u(L, t) = 0 \)
   IC: \( u(x, 0) = \begin{cases} \frac{hx}{\pi} & \text{if } 0 < x < a \\ \frac{h(L-x)}{L-a} & \text{if } a < x < L \end{cases} \), \( \frac{\partial u(x, 0)}{\partial t} = 0 \)

5) Solve the PDE \( u_{tt} = u_{xx} \) on \( 0 < x < \pi, \ 0 < t \), with
   BC: \( u(0, t) = u(\pi, t) = 0 \)
   IC: \( u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = x(\pi - x) \)

6) Solve the PDE \( u_{tt} = 12u_{xx} \) on \( 0 < x < 3, \ 0 < t \), with
   BC: \( u(0, t) = u(3, t) = 0 \)
   IC: \( u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = \sin(\pi x) \)

7) Solve the PDE \( u_{tt} = u_{xx} \) on \( 0 < x < \pi, \ 0 < t \), with
   BC: \( u(0, t) = u(\pi, t) = 0 \)
   IC: \( u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = \begin{cases} 0.1x & \text{if } 0 < x < \pi/2 \\ 0.1(\pi - x) & \text{if } \pi/2 < x < \pi \end{cases} \)

8) Solve the PDE \( u_{tt} = 4u_{xx} \) on \( 0 < x < 5, \ 0 < t \), with
   BC: \( u(0, t) = u(5, t) = 0 \)
   IC: \( u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 1 \)
Answers

1) \( u(x, t) = \sin(2x) \cos(4t) \)

2) \( u(x, t) = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^3} \left[ 1 - (-1)^n \right] \sin(n\pi x) \cos(n\pi t) \)
   \[ = \frac{8}{\pi^3} \left[ \sin(\pi x) \cos(\pi t) + \frac{1}{27} \sin(3\pi x) \cos(3\pi t) + \cdots \right] \]

3) \( u(x, t) = 5 \sin(\pi x) \cos\left(\frac{\pi t}{3}\right) - 3 \sin(2\pi x) \cos\left(\frac{2\pi t}{3}\right) \)

4) \( u(x, t) = \sum_{n=1}^{\infty} \frac{2hL^2}{n^2 \pi^2 a (L - a)} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \)

5) \( u(x, t) = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi} \left[ 1 - (-1)^n \right] \sin(nx) \sin(nt) \)
   \[ = \frac{8}{\pi} \left[ \sin(\pi x) \sin(\pi t) + \frac{1}{81} \sin(3\pi x) \sin(3\pi t) + \cdots \right] \]

6) \( u(x, t) = \frac{1}{2\pi \sqrt{3}} \sin(\pi x) \sin(2\pi \sqrt{3}t) \)

7) \( u(x, t) = \sum_{n=1}^{\infty} \frac{0.4}{n^3 \pi} \sin\left(\frac{n\pi}{2}\right) \sin(nx) \sin(nt) \)

8) \( u(x, t) = \sum_{n=1}^{\infty} \frac{5}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \sin\left(\frac{n\pi x}{5}\right) \sin\left(\frac{2n\pi t}{5}\right) \)
Chapter 13

Heat Equation

In this chapter, we will set up and solve heat equation. Although it is very similar to wave equation in form, the solutions will be quite different. We will generalize our methods to nonzero boundary conditions and two-dimensional problems.

13.1 Modeling Heat Flow

Figure 13.1: Heat Flow in One Dimension

Consider a long thin bar of length $L$ on $x$-axis. It has uniform density and cross section. The lateral surface is perfectly isolated, so the heat flow is in $x$-direction only. Experiments show that the amount of heat flow is proportional to the temperature gradient:

$$\frac{dQ}{dt} = -KS \frac{du}{dx} \quad (13.1)$$

where $Q$ is the heat, $u$ is the temperature, $S$ is the cross sectional area and $K$ is the thermal conductivity. The minus sign means that heat flows from higher to lower temperatures as we expect. A piece of the material of length $\Delta x$ has two neighbours, so the change in its temperature is determined by the net difference of heat flows:
CHAPTER 13. HEAT EQUATION

\[
\Delta Q = \left[ -KS \left. \frac{\partial u}{\partial x} \right|_x - \left( -KS \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} \right) \right] \Delta t
\]

\[
= \left[ \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x \right] KS \Delta t
\] (13.2)

We know that when a material receives heat, its temperature rises proportionally:

\[
\Delta Q = m\mu \Delta u
\]

\[
= S\Delta x \rho \mu \left( u|_{t+\Delta t} - u|_t \right)
\] (13.3)

where \(\mu\) is the specific heat and \(\rho\) is the density of the material. If we set these two \(\Delta Q\) values equal to each other, and rearrange, we will obtain

\[
KS \frac{\left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x}{\Delta x} = S\rho \mu \frac{u|_{t+\Delta t} - u|_t}{\Delta t}
\] (13.4)

In the limit \(\Delta x \to 0\) and \(\Delta t \to 0\) we will obtain second and first partial derivatives of \(u(x, t)\), so

\[
\frac{K}{\rho \mu} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}
\] (13.5)

If we define the diffusivity as \(k = K/(\rho \mu)\)

\[
[u_t = k u_{xx}]
\] (13.6)

This is the heat equation in one dimension. Its form is remarkably similar to wave equation, yet the solutions are different. This time, we will have only one Initial Condition \(u(x, 0) = f(x)\) which is the initial temperature distribution of the bar.

If the ends of the bar are kept at fixed temperatures, we have Boundary Conditions \(u(0, t) = T_1, u(L, t) = T_2\) where \(L\) is the length of the bar.

If the ends of the bar are isolated, the BC will be \(u_x(0, t) = u_x(L, t) = 0\)

A similar analysis shows that, in 2-dimensions, the heat equation is:

\[
[u_t = k(u_{xx} + u_{yy})]
\] (13.7)
13.2 Homogeneous Boundary Conditions

**Example 13.1** Solve the one dimensional heat equation \( u_t = ku_{xx} \) on a bar of length \( L \) with:

- **BC:** \( u(0, t) = u(L, t) = 0 \)
- **IC:** \( u(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ L - x & \text{if } \frac{L}{2} < x < L \end{cases} \)

Using separation of variables, we may write \( u(x, t) \) as

\[
    u(x, t) = F(x)G(t)
\]

Then \( FG' = kF''G \) or

\[
    \frac{G'}{kG} = \frac{F''}{F} = c
\]

This is possible only if both sides are equal to a constant. Therefore

\[
    \frac{G'}{kG} = \frac{F''}{F} = c
\]

Once again we have three cases. If \( c > 0 \), or \( c = 0 \), the solution is trivial. (Please verify!) Therefore

\[
    c < 0, c = -p^2, \quad \Rightarrow \quad F = A \cos px + B \sin px
\]

Using the BC we find \( A = 0 \) and

\[
    p = \frac{n\pi}{L}, \quad (n = 1, 2, 3\ldots)
\]

So

\[
    F_n = B_n \sin \frac{n\pi x}{L}, \\
    G' = -\frac{n^2\pi^2k}{L^2}G
\]

therefore

\[
    G_n = e^{-\lambda_n t} \quad \text{where} \quad \lambda_n = \frac{n^2\pi^2k}{L^2} \\
    u_n(x, t) = B_n \sin \frac{n\pi x}{L}e^{-\lambda_n t}
\]
and because of the superposition principle
\[ u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n t} \]

\( B_n \) can be determined as the Fourier sine coefficients of \( f(x) \). So
\[ B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \]

Performing the integration, (Please verify) we find
\[ B_n = \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2} \]

So the solution is
\[ u(x, t) = \frac{4L}{\pi^2} \left( \sin \frac{\pi x}{L} e^{-\lambda_1 t} - \frac{1}{3^2} \sin \frac{3\pi x}{L} e^{-\lambda_3 t} + \ldots \right) \]

**Example 13.2** Solve the PDE \( u_t = ku_{xx} \) with:

- **BC:** \( u_x(0, t) = u_x(L, t) = 0 \)
- **IC:** \( u(x, 0) = \cos \frac{\pi x}{L} \)

This is a bar with insulated ends. The solution is exactly the same as before up to the step
\[ F = A \cos px + B \sin px \quad \Rightarrow \quad F' = -Ap \sin(px) + Bp \cos(px) \]

Using the BC we find \( B = 0, \ Ap \sin(pL) = 0 \quad \Rightarrow \quad p = \frac{n\pi}{L} \)
\[ F_n = A_n \cos \frac{n\pi x}{L} \quad \Rightarrow \quad G_n(x, t) = \exp \left( -\frac{n^2\pi^2 kt}{L^2} \right) \]
\[ u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \exp \left( -\frac{n^2\pi^2 kt}{L^2} \right) \]

Using the IC we see that\[ A_n = \frac{2}{L} \int_0^L \cos \frac{\pi x}{L} \cos \frac{n\pi x}{L} \, dx \]

Using the orthogonality of trigonometric functions, we see that \( A_1 = 1 \) and all others are zero, so
\[ u(x, t) = \cos \frac{\pi x}{L} \exp \left( -\frac{\pi^2 kt}{L^2} \right) \]
13.3 Nonzero Boundary Conditions

**Steady State Solution:** The temperature distribution we get as $t \to \infty$ must be time independent. So we call it steady state solution.

We expect $\frac{\partial u}{\partial t} = 0$ which means $\frac{d^2u}{dx^2} = 0$ therefore the steady state solution must be

$$u(x) = Ax + B$$

**Example 13.3** Solve the steady state heat equation $u_t = ku_{xx}$ on $0 < x < L$ with BC: $u(0) = T_1$, $u(L) = T_2$

We know that $u(x) = Ax + B$ so

$$B = T_1, \ AL + T_1 = T_2$$

$$u(x) = \frac{T_2 - T_1}{L} x + T_1$$

**Example 13.4** Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < \pi, \ t > 0$$

with BC: $u(0, t) = 0, \ u(\pi, t) = 40, \ t > 0$

and IC: $u(x, 0) = 40, \ 0 < x < \pi$

First, we will find the steady state solution $u_1$. Obviously,

$$u_1(x) = \frac{40}{\pi} x$$

Now we will express the solution $u$ as a combination of two functions $u_1, u_2$. Here, $u_1$ is the steady state solution, and $u_2$ is the answer to a homogeneous BC problem:

$$u(x, t) = u_1(x) + u_2(x, t)$$

Let’s obtain the BC and IC for $u_2$
BC: \( u_2(0, t) = 0, \ u_2(\pi, t) = 0, \ t > 0 \)

IC: \( u_2(x, 0) = 40 \left( 1 - \frac{x}{\pi} \right), \ 0 < x < \pi \)

This is a new problem with homogeneous BC, so we can solve it as before.

\[
 u_2(x, t) = F(x)G(t)
\]

After similar steps,

\[
 F_n = B_n \sin nx
\]

and

\[
 G_n = e^{-n^2t}
\]

\[
 u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin nx e^{-n^2t}
\]

If we insert \( t = 0 \), we see that

\[
 u_2(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx = 40 \left( 1 - \frac{x}{\pi} \right)
\]

So, we can obtain \( B_n \) as the Fourier sine coefficients of the right hand side.

\[
 B_n = \frac{2}{\pi} \int_0^\pi 40 \left( 1 - \frac{x}{\pi} \right) \sin nx \, dx
\]

\[
 B_n = \frac{2}{\pi} \left[ - \frac{40 \cos nx}{n} - \frac{40}{\pi} \left( - \frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right) \right]_0^\pi
\]

\[
 = \frac{2}{\pi} \left[ - \frac{40}{n} \frac{1}{\pi} - \frac{40}{\pi} \left( - \pi \cos \frac{n\pi}{n} + 0 \right) \right]
\]

\[
 = \frac{2}{\pi} \left[ \frac{40}{n} \left( 1 - (-1)^n \right) + \frac{40}{\pi} (-1)^n \right]
\]

\[
 = \frac{80}{n\pi}
\]

\[
 u_2(x, t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} e^{-n^2t}
\]

Therefore the solution is

\[
 u(x, t) = \frac{40}{\pi} x + \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} e^{-n^2t}
\]
13.4 Two Dimensional Problems

We can generalize these methods to higher dimensions. Consider the temperature distribution on a rectangular plate of dimensions $2 \times 3$.

Example 13.5 Solve the PDE $u_t = k (u_{xx} + u_{yy})$ where $u = u(x, y, t)$ with:

**BC:** $u(0, y, t) = u(2, y, t) = 0$

$u(x, 0, t) = u(x, 3, t) = 0$

**IC:** $u(x, y, 0) = (4 - x^2)y(9 - y^2)$

This time we will apply the method of separation of variables to a three-variable function $u(x, y, t)$, therefore

$$u(x, y, t) = F(x)G(y)H(t)$$

After the usual steps, we obtain

$$F_n(x) = \sin \frac{n\pi x}{2}, \ G_m = \sin \frac{m\pi y}{3}, \ H_{nm} = A_{nm} \exp \left[-\left(\frac{n^2\pi^2}{4} + \frac{m^2\pi^2}{9}\right)kt\right]$$

Therefore

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{2} \sin \frac{m\pi y}{3} \exp \left[-\left(\frac{n^2\pi^2}{4} + \frac{m^2\pi^2}{9}\right)kt\right]$$

Using the initial condition

$$u(x, y, 0) = (4 - x^2)y(9 - y^2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{2} \sin \frac{m\pi y}{3}$$

$$A_{nm} = \frac{2}{\pi^6} \left( \int_0^2 x(4 - x^2) \sin \frac{n\pi x}{2} \, dx \right) \frac{2}{3} \left( \int_0^3 y(9 - y^2) \sin \frac{m\pi y}{3} \, dx \right)$$

$$= \left( \frac{96(-1)^{n+1}}{n^3\pi^3} \right) \left( \frac{324(-1)^{m+1}}{m^3\pi^3} \right)$$

$$u(x, y, t) = \frac{31104}{\pi^6} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{n^3\pi^3} \sin \frac{n\pi x}{2} \sin \frac{m\pi y}{3} \exp \left[-\left(\frac{n^2\pi^2}{4} + \frac{m^2\pi^2}{9}\right)kt\right]$$

The results are plotted on Figure 13.2 for three different $t$ values. We can easily see that $u \to 0$ as time increases.
Exercises

1) Solve the PDE \( u_t = u_{xx} \) on \( 0 < x < \pi, \ 0 < t \), with
   \[ BC: \ u(0, t) = u(\pi, t) = 0, \quad IC: \ u(x, 0) = \sin 2x \]

2) Solve the PDE \( u_t = 5u_{xx} \) on \( 0 < x < 4, \ 0 < t \), with
   \[ BC: \ u(0, t) = u(4, t) = 0, \quad IC: \ u(x, 0) = \sin \frac{\pi x}{2} - \sin \pi x \]

3) Solve the PDE \( u_t = ku_{xx} \) on \( 0 < x < L, \ 0 < t \), with
   \[ BC: \ u(0, t) = u(L, t) = 0, \quad IC: \ u(x, 0) = x(L - x) \]

4) Solve the PDE \( u_t = u_{xx} \) on \( 0 < x < \pi, \ 0 < t \), with
   \[ BC: \ u_x(0, t) = u_x(\pi, t) = 0, \quad IC: \ u(x, 0) = x \]

5) Solve the PDE \( u_t = 3u_{xx} \) on \( 0 < x < 10, \ 0 < t \), with
   \[ BC: \ u_x(0, t) = u_x(10, t) = 0, \quad IC: \ u(x, 0) = \cos 0.3\pi x \]

6) Solve the PDE \( u_t = ku_{xx} \) on \( 0 < x < L, \ 0 < t \), with
   \[ BC: \ u_x(0, t) = u_x(L, t) = 0, \quad IC: \ u(x, 0) = 1 - \frac{x}{L} \]

7) Solve the PDE \( u_t = u_{xx} \) with nonhomogeneous boundary conditions
   \[ BC : \ u(0, t) = 1, \ u(1, t) = 0, \quad IC: \ u(x, 0) = \sin(\pi x) \]

8) Solve the PDE \( u_t = ku_{xx} \) with nonhomogeneous boundary conditions
   \[ BC: \ u(0, t) = 0, \ u(L, t) = T, \quad IC: \ u(x, 0) = \begin{cases} 0 & \text{if } 0 < x < \frac{L}{2} \\ T & \text{if } \frac{L}{2} < x < L \end{cases} \]

9) Solve the PDE \( u_t = 8(u_{xx} + u_{yy}) \) on \( 0 < x < 2, \ 0 < y < 5, \ 0 < t \), with
   \[ BC: \ u(0, y, t) = u(2, y, t) = 0, \ u(x, 0, t) = u(x, 5, t) = 0 \]
   \[ IC: \ u(x, y, 0) = \sin \frac{\pi x}{2} \sin \frac{\pi y}{5} \]

10) Solve the PDE \( u_t = k(u_{xx} + u_{yy}) \) on \( 0 < x < a, \ 0 < y < b, \ 0 < t \), with
    \[ BC: \ u(0, y, t) = u(a, y, t) = 0, \ u(x, 0, t) = u(x, b, t) = 0 \]
    \[ IC: \ u(x, y, 0) = T \]
Answers

1) \( u(x,t) = \sin 2x e^{-4t} \)

2) \( u(x,t) = \frac{\pi x}{2} e^{-\frac{5}{2}\pi^2 t} - \sin(\pi x) e^{-5\pi^2 t} \)

3) \( u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3} [1 - (-1)^n] \sin \frac{n\pi x}{L} \exp \left( - \frac{n^2\pi^2 kt}{L^2} \right) \)

4) \( u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} [(-1)^n - 1] \cos n x e^{-n^2 t} \)

5) \( u(x,t) = \cos(0.3\pi x) e^{-0.2\pi^2 t} \)

6) \( u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} [1 - (-1)^n] \cos \frac{n\pi x}{L} \exp \left( - \frac{n^2\pi^2 kt}{L^2} \right) \)

7) \( u(x,t) = 1 - x + e^{-\pi^2 t} \sin \pi x - \frac{2}{\pi} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{n} e^{-n^2\pi^2 t} \)

8) \( u(x,t) = \frac{Tx}{L} + \sum_{n=1}^{\infty} \frac{2T}{n\pi} \cos \frac{n\pi}{2} \sin \frac{n\pi x}{L} e^{-n^2\pi^2 kL^2 \text{/} L^2} \)

\[= \frac{Tx}{L} - \frac{2T}{\pi} \left( \frac{1}{2} \sin \frac{2\pi x}{L} e^{-4\pi^2 kL^2 \text{/} L^2} - \frac{1}{4} \sin \frac{4\pi x}{L} e^{-16\pi^2 kL^2 \text{/} L^2} + \ldots \right) \]

9) \( u(x,y,t) = \sin \frac{\pi x}{2} \sin \frac{\pi y}{5} e^{-2.32\pi^2 t} \)

10) \( u(x,y,t) = \frac{4T}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-k\pi^2 \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) t} \)

Where \( A_{nm} = \frac{(1 - (-1)^n)(1 - (-1)^m)}{nm} \)
Chapter 14

Laplace Equation

Laplace equation is the last PDE we will consider. It is different from the wave and heat equations in that, time is not a variable. We can also think of Laplace equation as the equilibrium configuration of heat and wave equations. It is possible to express these equations in any coordinate system that suits the geometry of the problem. As an example, we will consider polar coordinates in this chapter.

14.1 Rectangular Coordinates

Laplace equation in two dimensions is

\[ u_{xx} + u_{yy} = 0 \]  \hspace{1cm} (14.1)

where \( u = u(x, y) \). The potential function for gravitational force in free space satisfies Laplace equation. Similarly, the electrostatic potential also satisfies the same equation. Therefore Laplace equation is sometimes called Potential Equation.

There are no time derivatives in Laplace Equation, therefore there are no initial conditions. We just have the boundary conditions. If the values of \( u \) are given on the boundary, the problem is called a Dirichlet problem, if the values of the normal derivative are given on boundary, it is called a Neumann problem. It is also possible to set up mixed problems. In this book, we will only consider Dirichlet problems.
Let’s consider a Dirichlet problem on the rectangle shown in Figure 14.1.

\[ u_{xx} + u_{yy} = 0 \quad \text{on} \quad 0 < x < a, \quad 0 < y < b \]  

(14.2)

with BC:

\[ u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = f(x) \]  

(14.3)

Using the method of separation of variables, we start with the assumption 
\[ u(x, y) = F(x)G(y) \]  
and inserting in equation, we obtain

\[ \frac{F''}{F} = -\frac{G''}{G} = k \]  

(14.4)

Depending on the sign of \( k \), we have three different cases:

**Case 1)** \( k = 0 \), \( u = (Ax + B)(Cy + D) \),

**Case 2)** \( k > 0 \), \( k = p^2 \), \( u = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \),

**Case 3)** \( k < 0 \), \( k = -p^2 \), \( u = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \).

Using the BC \( x = 0 \Rightarrow u = 0 \) and \( x = a \Rightarrow u = 0 \) we can easily see that the first two cases give trivial solutions. Using the same conditions on the third case, we obtain \( A = 0, p = \frac{n\pi}{a} \) as we did in the previous chapters.

\[ u_n(x, y) = B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \]  

(14.5)

The third BC \( y = 0 \Rightarrow u = 0 \) gives

\[ C + D = 0 \Rightarrow D = -C \]  

(14.6)

Remember the hyperbolic sine function, which is defined as

\[ \sinh y = \frac{e^y - e^{-y}}{2} \]  

(14.7)

Now we can express the solution in terms of trigonometric and hyperbolic functions as:

\[ u_n(x, y) = B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \]  

(14.8)
Superposition of these solutions give

\[ u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \]  

(14.9)

We have only the fourth boundary condition left: \( y = b \Rightarrow u = f(x) \)

\[ u(x, b) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} = f(x) \]  

(14.10)

Obviously, \( B_n \sinh \frac{n\pi b}{a} \) are the Fourier sine coefficients of \( f(x) \), so

\[ B_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_{0}^{a} f(x) \sin \frac{n\pi x}{a} \, dx \]  

(14.11)

**Remark:** If two sides have nonzero BC, we can consider them as two separate problems having zero BC on 3 sides, find the solutions and then add them to obtain the result, as you can see on Figure 14.2.

**Figure 14.2:** Nonzero Boundary Conditions on two sides
CHAPTER 14. LAPLACE EQUATION

Example 14.1  Solve $u_{xx} + u_{yy} = 0$ on $0 < x < 2, 0 < y < 1,$ with
BC: $u(0, y) = 0, u(2, y) = 0, u(x, 0) = 0, u(x, 1) = 1$

Using the steps above, we find

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \sinh \frac{n\pi y}{2}$$

where

$$B_n \sinh \frac{n\pi}{2} = \int_{0}^{2} \sin \frac{n\pi x}{2} \, dx$$

$$B_n = \frac{2[1 - (-1)^n]}{n\pi \sinh \frac{n\pi}{2}}$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi \sinh \frac{n\pi}{2}} \sin \frac{n\pi x}{2} \sinh \frac{n\pi y}{2}$$

You can see the solution on Figure 14.3 (up).

Example 14.2  Solve $u_{xx} + u_{yy} = 0$ on $0 < x < 1, 0 < y < 1,$ with
BC: $u(x, 0) = 0, u(x, 1) = 0, u(0, y) = 0, u(1, y) = 3y(1 - y)$

The solution satisfying the first three boundary conditions is:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x) \sin(n\pi y)$$

Inserting $x = 1$ and using the fourth boundary condition, we obtain

$$\sinh(n\pi) \, c_n = 2 \int_{0}^{1} 3y(1 - y) \sin(n\pi y) \, dy$$

$$\sinh(n\pi) \, c_n = 6 \left[ -\frac{y \cos n\pi y}{n\pi} + \frac{\sin n\pi y}{n^2\pi^2} + \frac{y^2 \cos n\pi y}{n\pi} - \frac{2y \sin n\pi y}{n^2\pi^2} - \frac{2 \cos n\pi y}{n^3\pi^3} \right]_{0}^{1}$$

$$c_n = \frac{12[1 - (-1)^n]}{n^3\pi^3 \sinh(n\pi)}$$

$$u(x, y) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^3 \sinh(n\pi) \sin(n\pi x) \sin(n\pi y)}$$

Figure 14.3 (down) gives the plot.
Figure 14.3: Solution of the Dirichlet Problem
14.2 Polar Coordinates

If the region of interest is circular, we have to express the Laplace Equation in polar coordinates to be able to use the boundary conditions.

We will start with \( x = r \cos \theta \), \( y = r \sin \theta \) and use chain rule to express the derivatives of \( u \) with respect to \( r \) and \( \theta \).

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \tag{14.12}
\]

\[
r^2 = x^2 + y^2 \tag{14.13}
\]

\[
2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \tag{14.14}
\]

If you complete this derivation, (which is a nice exercise in calculus) you will obtain the Laplace equation in polar coordinates:

\[
u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta \theta}}{r^2} = 0 \tag{14.15}\]

To solve the Laplace equation inside a circle of radius \( a \) together with the boundary condition \( u(a, \theta) = f(\theta) \), we start the method of separation of variables with the assumption \( u(r, \theta) = F(r)G(\theta) \).

Inserting this in (14.15) we obtain

\[
F''G + \frac{F'G}{r} + \frac{FG''}{r^2} = 0 \tag{14.16}
\]

\[
\frac{r^2F''}{F} + \frac{rF'}{F} = - \frac{G''}{G} = k \tag{14.17}
\]

where \( k \) is the separation constant. Once again we have three possibilities:

**Case 1** \( k = 0 \), \( u = (A \ln r + B)(C \theta + D) \),

**Case 2** \( k > 0 \), \( k = p^2 \), \( u = (Ar^p + Br^{-p})(C \cos p\theta + D \sin p\theta) \),

**Case 3** \( k < 0 \), \( k = -p^2 \), \( u = [A \cos(p \ln r) + B \sin(p \ln r)](Ce^{p\theta} + De^{-p\theta}) \).

We expect the solution to be periodic in \( \theta \) with period \( 2\pi \). Case 3 does not satisfy this, so we eliminate this case.
In Case 1, we have to choose \( C = 0 \) for periodicity. Besides, \( \ln r \) is undefined at \( r = 0 \). So \( A = 0 \). Therefore the contribution of Case 1 is only a constant.

In Case 2, \( r^{-p} \) is undefined at \( r = 0 \), so we choose \( B = 0 \). The resulting separated solution is:

\[
  u_n(r, \theta) = r^n (C_n \cos n\theta + D_n \sin n\theta) \quad (14.18)
\]

Note that \( n \) must be an integer for periodicity.

After superposition, we obtain the general solution as

\[
  u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta) \quad (14.19)
\]

The boundary condition is: \( u(a, \theta) = f(\theta) \), we can find \( C_n \) and \( D_n \) using the Fourier expansion of \( f \).

\[
  C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \\
  C_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \\
  D_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \quad (14.20)
\]

**Remark:** If the region is outside the circle, the same ideas apply. We have to eliminate \( \ln r \) because it is not finite at infinity. The only difference is that we should have the negative powers of \( r \), because they will be bounded as \( r \to \infty \). So

\[
  u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^{-n} (C_n \cos n\theta + D_n \sin n\theta) \quad (14.21)
\]

**Remark:** If we have a region between two circles as \( a < r < b \), we need both the positive and negative powers of \( r \) as well as the logarithmic term.
Example 14.3  Solve Laplace equation in the region $0 \leq r < 5$, with

BC: $u(5, \theta) = \begin{cases} -1 & \text{if } -\pi < \theta < 0 \\ 1 & \text{if } 0 < \theta < \pi \end{cases}$

We know that the general solution in this case is

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$$

The boundary condition gives

$$u(5, \theta) = C_0 + \sum_{n=1}^{\infty} 5^n (C_n \cos n\theta + D_n \sin n\theta) = f(\theta)$$

The Fourier coefficients of $f$ are

$$C_0 = 0, \quad C_n = 0, \quad D_n = \frac{2}{n\pi 5^n} [1 - (-1)^n]$$

$$u(r, \theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{[1 - (-1)^n]} \left(\frac{r}{5}\right)^n \frac{\sin n\theta}{n}$$

The solution is plotted on Figure 14.5 (up).

Example 14.4  Solve Laplace equation in the region $0 \leq r < 2$, with

BC: $u(2, \theta) = \sin(3\theta)$

Inserting $r = 2$ in the solution

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$$

we obtain

$$u(2, \theta) = C_0 + \sum_{n=1}^{\infty} 2^n (C_n \cos n\theta + D_n \sin n\theta) = \sin 3\theta$$

We can easily see that the only nonzero Fourier coefficient is $D_3$

$$2^3 D_3 = 1 \quad \Rightarrow \quad D_3 = \frac{1}{8}$$

$$u(r, \theta) = \frac{1}{8} r^3 \sin 3\theta$$

The solution is plotted on Figure 14.5 (down).
Figure 14.5: Potential on a Circle
Example 14.5  Solve Laplace equation in the region $3 \leq r$, with
BC: $u(3, \theta) = \cos^2 \theta$

This time the region is outside the circle so the general solution is

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^{-n} (C_n \cos n\theta + D_n \sin n\theta)$$

The boundary condition gives

$$u(3, \theta) = C_0 + \sum_{n=1}^{\infty} 3^{-n} (C_n \cos n\theta + D_n \sin n\theta) = \cos^2 \theta$$

We know that $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, so

$$u(r, \theta) = \frac{1}{2} \left( 1 + \frac{9}{r^2} \cos 2\theta \right)$$

Example 14.6  Solve Laplace equation in the region $1 \leq r \leq 2$, with
BC: $u(1, \theta) = 5 \sin 3\theta$,  $u(2, \theta) = 3 \ln 2 + 40 \sin 3\theta$

The region is between two circles, so the general solution is

$$u(r, \theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) + \sum_{n=1}^{\infty} r^{-n} (C_n \cos n\theta + D_n \sin n\theta)$$

We can directly see that all the coefficients except $A_0, B_0, B_3, D_3$ must be zero, therefore

$$u(r, \theta) = A_0 + B_0 \ln r + B_3 \sin 3\theta r^3 + D_3 \frac{\sin 3\theta}{r^3}$$

Using the boundary conditions at $r = 1$ and $r = 2$, we obtain $A_0 = 0, B_0 = 3, B_3 = 5, D_3 = 0$, so

$$u(r, \theta) = 3 \log r + 5r^3 \sin 3\theta$$

Remark: We will state without proof that if $u$ satisfies Laplace equation in a region, then its value at any point is equal to the average values around any circle (within that region).

Using this principle, we can easily derive the result that maximum and minimum values of $u$ must occur on the boundary.

The given solution plots illustrate these principles.
Exercises

1) Solve the PDE $u_{xx} + u_{yy} = 0$, on $0 < x < 2$, $0 < y < 2$, with
   
   BC: $u(x, 0) = 0$, $u(x, 2) = 0$, $u(0, y) = 0$, $u(2, y) = \sin \frac{3\pi y}{2}$

2) Solve the PDE $u_{xx} + u_{yy} = 0$, on $0 < x < 5$, $0 < y < 1$, with
   
   BC: $u(x, 0) = \sin \pi x$, $u(x, 1) = 0$, $u(0, y) = 0$, $u(5, y) = 0$

3) Solve the PDE $u_{xx} + u_{yy} = 0$, on $0 < x < 2$, $0 < y < 8$, with
   
   BC: $u(x, 0) = 0$, $u(x, 8) = 0$, $u(0, y) = 0$, $u(2, y) = \begin{cases} 1 & \text{if } 0 < y < 4 \\ -1 & \text{if } 4 < y < 8 \end{cases}$

4) Solve the PDE $u_{xx} + u_{yy} = 0$, on $0 < x < 2$, $0 < y < 2$, with
   
   BC: $u(x, 0) = 0$, $u(x, 2) = \sin \frac{\pi x}{2}$, $u(0, y) = 0$, $u(2, y) = \sin \frac{\pi y}{2}$

5) Solve the PDE $u_{xx} + u_{yy} = 0$, on $0 < x < 3$, $0 < y < 2$, with
   
   BC: $u(x, 0) = 0$, $u(x, 2) = 0$, $u(0, y) = \sin \frac{5\pi y}{2}$, $u(3, y) = \sin \frac{7\pi y}{2}$

6) Solve the PDE $u_{rr} + \frac{u_r}{r} + \frac{u_\theta\theta}{r^2} = 0$ on $0 \leq r < 1$, with
   
   BC: $u(1, \theta) = \cos 4\theta$

7) Solve the PDE $u_{rr} + \frac{u_r}{r} + \frac{u_\theta\theta}{r^2} = 0$ on $0 \leq r < 4$, with
   
   BC: $u(4, \theta) = 2\sin 2\theta - 7\cos 3\theta$

8) Solve the PDE $u_{rr} + \frac{u_r}{r} + \frac{u_\theta\theta}{r^2} = 0$ on $3 < r$, with
   
   BC: $u(3, \theta) = 5 - 5\cos 3\theta$

9) Solve the PDE $u_{rr} + \frac{u_r}{r} + \frac{u_\theta\theta}{r^2} = 0$ on $3 < r < 5$, with
   
   BC: $u(3, \theta) = 4$, $u(5, \theta) = 12$

10) Solve the PDE $u_{rr} + \frac{u_r}{r} + \frac{u_\theta\theta}{r^2} = 0$ on $2 < r < 3$, with
    
    BC: $u(2, \theta) = -5\sin 2\theta$, $u(3, \theta) = 10\cos 2\theta$
Answers

1) \( u(x, y) = \frac{1}{\sinh 3\pi} \sinh \frac{3\pi x}{2} \sin \frac{3\pi y}{2} \)

2) \( u(x, y) = \frac{1}{\sinh \pi} \sin \pi x \sinh \pi (1 - y) \)

3) \( u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^n - 2 \cos \frac{n\pi}{2}}{n \sinh \frac{n\pi}{4}} \sinh \frac{n\pi x}{8} \sin \frac{n\pi y}{8} \)

4) \( u(x, y) = \frac{1}{\sinh \pi} \left( \sin \frac{\pi x}{2} \sinh \frac{\pi y}{2} + \sin \frac{\pi y}{2} \sinh \frac{\pi x}{2} \right) \)

5) \( u(x, y) = \frac{1}{\sinh \frac{2\pi}{2}} \sin \frac{7\pi y}{2} \sinh \frac{7\pi x}{2} - \frac{1}{\sinh \frac{15\pi}{2}} \sin \frac{5\pi y}{2} \sinh \frac{5\pi(x - 3)}{2} \)

6) \( u(r, \theta) = r^4 \cos 4\theta \)

7) \( u(r, \theta) = 2 \left( \frac{r}{4} \right)^2 \sin 2\theta - 7 \left( \frac{r}{4} \right)^3 \cos 3\theta \)

8) \( u(r, \theta) = 5 - 5 \left( \frac{3}{r} \right)^3 \cos 3\theta \)

9) \( u(r, \theta) = \frac{4 \ln 5 - 12 \ln 3 + 8 \ln r}{\ln 5 - \ln 3} \)

10) \( u(r, \theta) = \frac{9}{13} \left( 2r^2 - \frac{32}{r^2} \right) \cos 2\theta + \frac{4}{13} \left( r^2 - \frac{81}{r^2} \right) \sin 2\theta \)
To the Student

If you have reached this point after solving all (or most) of the exercises, you must have covered a lot of ground. But there’s no end to differential equations. This was just a brief introduction. For further study, you may consult the books listed in the references.

[6, 8] and [9] are big and useful books that contain all topics covered here and many other ones besides.

For ordinary differential equations, [2, 11, 12, 14] give a complete treatment with a large number of exercises.

For partial differential equations, [1] and [7] are good introductory books that illustrate main ideas.

Detailed information on Fourier Series can be found on [3].

There are many aspects of differential equations that we did not even touch in this book.

For a history of this subject, you may consult [13].

For nonlinear equations and dynamical systems, which is a vast subject requiring another book even for the introduction, [10] and [15] will be a good starting point.

For numerical methods, you may read the relevant chapters of [4] and [5].
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Corrections of
*Lecture Notes on Differential Equations*
by Emre Sermutlu

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<td>23</td>
<td>14</td>
<td>( y(\pi) = 0, \ y(-\pi) = 0 )</td>
<td>( y(0) = 0, \ y'(0) = 1 )</td>
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<td>85</td>
<td>1</td>
<td>( 2\pi(\sin x - \cdots) )</td>
<td>( 2(\sin x - \cdots) )</td>
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<td>11.2 Result</td>
<td>( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} )</td>
<td>( 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} )</td>
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